The Blueprint For Formalizing Geometric Algebra in Lean

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Introduction

The goal of this document is to provide a detailed account of the formalization of Geometric Algebra (GA) a.k.a. Clifford Algebra [Hestenes and Sobczyk(1984)] in the Lean 4 theorem prover and programming language [Moura and Ullrich(2021), de Moura et al.(2015), Ullrich(2023)] and using its Mathematical Library Mathlib [The mathlib Community(2020)].

The web version of this blueprint is available here.

1 Preliminaries

This section introduces the algebraic environment of Clifford Algebra, covering vector spaces, groups, algebras, representations, modules, multilinear algebras, quadratic forms, filtrations and graded algebras.

The material in this section should be familiar to the reader, but it is worth reading through it to become familiar with the notation and terminology that is used, as well as their counterparts in Lean, which usually require some additional treatment, both mathematically and technically (probably applicable to other formal proof verification systems).

Details can be found in the references in corresponding section, or you may hover a definition/theorem, then click on L N for the Lean 4 code.

In this section, we follow [Jadczyk(2019)], with supplements from [Garling(2011), Chen(2016)], and modifications to match the counterparts in Lean's Mathlib .

Remark 1.0.1 — We unify the informal mathematical language for a definition to:

Let A be a `concept` A. A `concept` B is a set/pair/triple/tuple $(B, \mathrm{op}, \ldots)$, satisfying:

1. B is a `concept` C over A under op .
2. `formula` for all `elements in` B (`property`).
3. for each `element` in `concept` A there exists `element` such that `formula` for all `elements in concept` B.
4. op is called `op name`, for all `elements in` B, we have
   (i) `formula`
   (ii) `formula`

   (`property`).

By default, A is a set, op is a binary operation on A.

1.1 Basics: from groups to modules

Definition 1.1.1 (Group). A **group** is a pair $(G, *)$, satisfying:

1. $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$ (**associativity**).
2. there exists $1 \in G$ such that $1 * a = a * 1 = a$ for all $a \in G$.
3. for each $a \in G$ there exists $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = 1$.

Remark 1.1.2 — It then follows that e, the **identity element**, is unique, and that for each $g \in G$ the **inverse** g^{-1} is unique.

A group G is abelian, or **commutative**, if $g * h = h * g$ for all $g, h \in G$.

Remark 1.1.3 — In literatures, the binary operation are usually denoted by juxtaposition, and is understood to be a mapping $(g, h) \mapsto g * h$ from $G \times G$ to G.

Mathlib uses a slightly different way to encode this, $G \to G \to G$ is understood to be $G \to (G \to G)$, that sends $g \in G$ to a mapping that sends $h \in G$ to $g * h \in G$.

Furthermore, a mathematical construct is represented by a "type", as Lean has a dependent type theory foundation, see [Carneiro(2019)] and [Ullrich(2023), section 3.2].

It can be denoted multiplicatively as $*$ in Group or additively as $+$ in AddGroup, where e will be denoted by 1 or 0, respectively.

Sometimes we use notations with subscripts (e.g. $*_G$, 1_G) to indicate where they are.

We will use the corresponding notation in Mathlib for future operations without further explanation.

Definition 1.1.4 (Monoid). A **monoid** is a pair $(R, *)$, satisfying:

1. $(a * b) * c = a * (b * c)$ for all $a, b, c \in R$ (**associativity**).
2. there exists an element $1 \in R$ such that $1 * a = a * 1 = a$ for all $a \in R$ i.e. 1 is the multiplicative identity (**neutral element**).

Definition 1.1.5 (Ring). A **ring** is a triple $(R, +, *)$, satisfying:

1. R is a **commutative group** under $+$.
2. R is a **monoid** under $*$.
3. for all $a, b, c \in R$, we have

   (i) $a * (b + c) = a * b + a * c$

   (ii) $(a + b) * c = a * c + b * c$

   (left and right **distributivity** over $+$).

Remark 1.1.6 — In applications to Clifford algebras R will be always assumed to be **commutative**.

Definition 1.1.7 (Division ring). A **division ring** is a ring $(R, +, *)$, satisfying:

1. R contains at least 2 elements.
2. for all $a \neq 0$ in R, there exists a multiplicative inverse $a^{-1} \in R$ such that

$$a * a^{-1} = a^{-1} * a = 1$$

Definition 1.1.8 (Module). Let R be a commutative ring. A **module** over R (in short R**-module**) is a pair $(M, \bullet)$, satisfying:

1. M is a group under +.
2. $\bullet : R \to M \to M$ is called **scalar multiplication**, for every $a, b \in R$, $x, y \in M$, we have
   (i) $a \bullet (x + y) = a \bullet x + b \bullet y$
   (ii) $(a + b) \bullet x = a \bullet x + b \bullet x$
   (iii) $a * (b \bullet x) = (a * b) \bullet x$
   (iv) $1_R \bullet x = x$

Remark 1.1.9 — The notation of scalar multiplication is generalized as heterogeneous scalar multiplication HMul in Mathlib :

$$\bullet : \alpha \to \beta \to \gamma$$

where α, β, γ are different types.

Definition 1.1.10 (Vector space)**.** If R is a **division ring**, then a module M over R is called a **vector space**.

Remark 1.1.11 — For generality, Mathlib uses Module throughout for vector spaces, particularly, for a vector space V, it's usually declared as

```
/--
    Let K be a division ring, a module V over K is a vector space
    where being a module requires V to be a commutative group
    over +.
-/
variable [DivisionRing K] [AddCommGroup V] [Module K V]
```

for definitions/theorems about it, and most of them can be found under Mathlib.LinearAlgebra e.g. LinearIndependent.

Remark 1.1.12 — A **submodule** N of M is a module N such that every element of N is also an element of M.

If M is a vector space then N is called a **subspace**.

Definition 1.1.13 (Dual module)**.** The **dual module** $M^* : M \to_{l[R]} R$ is the R-module of all linear maps from M to R.

1.2 Algebras

Definition 1.2.1 (Ring homomorphism)**.** Let $(\alpha, +_\alpha, *_\alpha)$ and $(\beta, +_\beta, *_\beta)$ be rings.

A **ring homomorphism** from α to β is a map $\mathit{1} : \alpha \to_{+*} \beta$ such that

(i) $\mathit{1}(x +_\alpha y) = \mathit{1}(x) +_\beta \mathit{1}(y)$ for each $x, y \in \alpha$.

(ii) $\mathit{1}(x *_\alpha y) = \mathit{1}(x) *_\beta \mathit{1}(y)$ for each $x, y \in \alpha$.

(iii) $\mathit{1}(1_\alpha) = 1_\beta$.

> **Remark 1.2.2** — **Isomorphism** $A \cong B$ is a bijective **homomorphism** $\phi : A \to B$ (it follows that $\phi^{-1} : B \to A$ is also a **homomorphism**).
>
> **Endomorphism** is a **homomorphism** from an object to itself, denoted $\mathrm{End}(A)$.
>
> **Automorphism** is an **endomorphism** which is also an **isomorphism**, denoted $\mathrm{Aut}(A)$.

Definition 1.2.3 (Algebra). Let R be a commutative ring. An **algebra** A over R is a pair $(A, \bullet)$, satisfying:

1. A is a **ring** under $*$.
2. there exists a **ring homomorphism** from R to A, denoted $\mathit{1} : R \to_{+*} A$.
3. $\bullet : R \to M \to M$ is a **scalar multiplication**
4. for every $r \in R$, $x \in A$, we have
   (i) $r * x = x * r$ (commutativity between R and A)
   (ii) $r \bullet x = r * x$ (definition of scalar multiplication)

where we omitted that the ring homomorphism $\mathit{1}$ is applied to r before each multiplication.

> **Remark 1.2.4** — Following literatures, for $r \in R$, usually we write $\mathit{1}_A(r) : R \to_{+*} A$ as a product $r\mathit{1}_A$ if not omitted, while they are written as a call to `algebraMap _ _ r` in Mathlib , which is defined to be `Algebra.toRingHom r`.

> **Remark 1.2.5** — The definition above (adopted in Mathlib) is more general than the definition in literature:
>
> Let R be a commutative ring. An **algebra** A over R is a pair $(M, *)$, satisfying:
>
> 1. A is a **module** M over R under $+$ and $\bullet$.
> 2. A is a **ring** under $*$.
> 3. For $x, y \in A, a \in R$, we have
>
> $$a \bullet (x * y) = (a \bullet x) * y = x * (a \bullet y)$$
>
> See *Implementation notes* in Algebra for details.

Remark 1.2.6 — What's simply called algebra is actually associative algebra with identity, a.k.a. **associative unital algebra**. See the red herring principle for more about such phenomenon for naming, particularly the example of (possibly) **nonassociative algebra**.

Definition 1.2.7 (Algebra homomorphism). Let A and B be R-algebras. 1_A and 1_B are **ring homomorphisms** from R to A and B, respectively.

A **algebra homomorphism** from A to B is a map $f : \alpha \to_a \beta$ such that

1. f is a **ring homomorphism**
2. $f(1_A(r)) = 1_B(r)$ for each $r \in R$

Definition 1.2.8 (Ring quotient). Let R be a non-commutative ring, r an arbitrary equivalence relation on R. The **ring quotient** of R by r is the quotient of R by the strengthen equivalence relation of r such that for all a, b, c in R:

1. $a + c \sim b + c$ if $a \sim b$
2. $a * c \sim b * c$ if $a \sim b$
3. $a * b \sim a * c$ if $b \sim c$

i.e. the equivalence relation is compatible with the ring operations $+$ and $*$.

Remark 1.2.9 — As ideals haven't been formalized for the non-commutative case, Mathlib uses RingQuot in places where the quotient of non-commutative rings by ideal is needed.

The universal properties of the quotient are proven, and should be used instead of the definition that is subject to change.

Definition 1.2.10 (Free algebra). Let X be an arbitrary set. An **free R-algebra** A on X (or "**generated by** X") is the **ring quotient** of the following inductively constructed set $A_{\mathtt{pre}}$

1. for all x in X, there exists a map $X \to A_{\mathtt{pre}}$.
2. for all r in R, there exists a map $R \to A_{\mathtt{pre}}$.
3. for all a, b in $A_{\mathtt{pre}}$, $a + b$ is in $A_{\mathtt{pre}}$.
4. for all a, b in $A_{\mathtt{pre}}$, $a * b$ is in $A_{\mathtt{pre}}$.

by that equivalence relation that makes A an **R-algebra**, namely:

1. there exists a **ring homomorphism** from R to $A_{\mathtt{pre}}$, denoted $R \to_{+*} A_{\mathtt{pre}}$.
2. A is a **commutative group** under $+$.
3. A is a **monoid** under $*$.
4. left and right **distributivity** under $*$ over $+$.

5. $0 * a \sim a * 0 \sim 0$.

6. for all a, b, c in A, if $a \sim b$, we have

   (i) $a + c \sim b + c$
   (ii) $c + a \sim c + b$
   (iii) $a * c \sim b * c$
   (iv) $c * a \sim c * b$

   (**compatibility** with the ring operations $+$ and $*$)

Remark 1.2.11 — What we defined here is the **free (associative, unital) R-algebra** on X, it can be denoted $R\langle X\rangle$, expressing that it's freely generated by R and X, where X is the set of generators.

Definition 1.2.12 (Linear map)**.** Let R, S be rings, M an R-module, N an S-module. A **linear map** from M to N is a function $f : M \to_l N$ over a ring homomorphism $\sigma : R \to_{+*} S$, satisfying:

1. $f(x + y) = f(x) + f(y)$ for all $x, y \in M$.
2. $f(r \bullet x) = \sigma(r) \bullet f(x)$ for all $r \in R$, $x \in M$.

Remark 1.2.13 — The set of all linear maps from M to M' is denoted $\mathrm{Lin}(M, M')$, and $\mathrm{Lin}(M)$ for mapping from M to itself.

$\mathrm{Lin}(M)$ is an **endomorphism**.

Definition 1.2.14 (Tensor algebra)**.** Let A be a **free R-algebra** generated by module M, let $\iota : M \to A$ denote the map from M to A.

An **tensor algebra** over M (or "of M") T is the **ring quotient** of the **free R-algebra** generated by M, by the equivalence relation satisfying:

1. for all a, b in M, $\iota(a + b) \sim \iota(a) + \iota(b)$.
2. for all r in R, a in M, $\iota(r \bullet a) \sim r * \iota(a)$.

i.e. making the inclusion of M into an **R-linear map**.

Remark 1.2.15 — The definition above is equivalent to the following definition in literature:

Let M be a module over R. An algebra T is called a **tensor algebra** over M (or "of M") if it satisfies the following universal property

1. T is an algebra containing M as a **submodule**, and it is **generated by** M,
2. Every linear mapping λ of M into an algebra A over R, can be extended to a **homomorphism** θ of T into A.

1.3 Forms

Definition 1.3.1 (Bilinear form)**.** Let R be a ring, M an R-module. An **bilinear form** B over M is a map $B : M \to M \to R$, satisfying:

1. $B(x + y, z) = B(x, z) + B(y, z)$
2. $B(x, y + z) = B(x, y) + B(x, z)$
3. $B(a \bullet x, y) = a * B(x, y)$
4. $B(x, a \bullet y) = a * B(x, y)$

for all $a \in R, x, y, z \in M$.

Definition 1.3.2 (Quadratic form)**.** Let R be a commutative ring, M a R-module. An **quadratic form** Q over M is a map $Q : M \to R$, satisfying:

1. $Q(a \bullet x) = a * a * Q(x)$ for all $a \in R, x \in M$.
2. there exists a companion **bilinear form** $B : M \to M \to R$, such that $Q(x + y) = Q(x) + Q(y) + B(x, y)$

In some literatures, the bilinear form is denoted Φ, and called the **polar form** associated with the quadratic form Q, or simply the polar form of Q.

> **Remark 1.3.3** — This notion generalizes to commutative semirings using the approach in [Izhakian et al.(2016)].

2 Clifford Algebra

2.1 Definition

Let M be a module over a commutative ring R, equipped with a quadratic form $Q : M \to R$.

Definition 2.1.1 (Clifford algebra)**.** Let $\iota : M \to_{l[R]} T(M)$ be the canonical R-linear map for the tensor algebra $T(M)$.

Let $1 : R \to_{+*} T(M)$ be the canonical map from R to $T(M)$, as a ring homomorphism.

A **Clifford algebra** over Q, denoted $C\ell(Q)$, is the **ring quotient** of the **tensor algebra** $T(M)$ by the equivalence relation satisfying $\iota(m)^2 \sim 1(Q(m))$ for all $m \in M$.

The natural quotient map is denoted $\pi : T(M) \to C\ell(Q)$ in some literatures, or π_Φ/π_Q to emphasize the bilinear form Φ or the quadratic form Q, respectively.

Remark 2.1.2 — In literatures, M is often written V, and the quotient is taken by the two-sided ideal I_Q generated from the set $\{v \otimes v - Q(v) \mid v \in V\}$.

As of writing, Mathlib does not have direct support for two-sided ideals, but it does support the equivalent operation of taking the **ring quotient** by a suitable closure of a relation like $v \otimes v \sim Q(v)$.

Hence the definition above.

Remark 2.1.3 — This definition and what follows in Mathlib is initially presented in [Wieser and Song(2022)], some further developments are based on [Grinberg(2016)], and in turn based on [Bourbaki(2007)] which is in French and never translated to English.

The most informative English reference on [Bourbaki(2007)] is [Jadczyk(2019)], which has an updated exposition in [Jadczyk(2023)].

Example 2.1.4 (Clifford algebra over a vector space)

Let V be a vector space $\mathbb{R}^n$ over $\mathbb{R}$, equipped with a quadratic form Q.

Since $\mathbb{R}$ is a commutative ring and V is a module, definition 2.1.1 of Clifford algebra applies.

Definition 2.1.5 (Clifford map)**.** We denote the canonical R-linear map to the Clifford algebra $C\ell(M)$ by $\iota : M \to_{l[R]} C\ell(M)$.

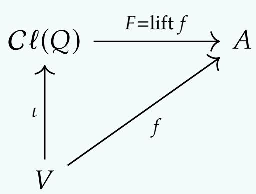
It's denoted i_Φ or just i in some literatures.

Definition 2.1.6 (Clifford lift)**.** Given a linear map $f : M \to_{l[R]} A$ into an R-algebra A, satisfying $f(m)^2 = Q(m)$ for all $m \in M$, called **is Clifford**, the canonical **lift** of f is defined to be a **algebra homomorphism** from $C\ell(Q)$ to A, denoted $\operatorname{lift} f : C\ell(Q) \to_a A$.

Theorem 2.1.7 (Universal property)

Given $f : M \to_{l[R]} A$, which **is Clifford**, $F = \operatorname{lift} f$ (denoted $\bar{f}$ in some literatures), we have:

1) $F \circ \iota = f$, i.e. the following diagram commutes:



2) lift is unique, i.e. given $G : C\ell(Q) \to_a A$, we have:
$G \circ \iota = f \iff G = \operatorname{lift} f$

Remark 2.1.8 — The universal property of the Clifford algebras is now proven, and should be used instead of the definition that is subject to change.

Definition 2.1.9 (Exterior algebra)**.** An **Exterior algebra** over M is the Clifford algebra $C\ell(Q)$ where $Q(m) = 0$ for all $m \in M$.

2.2 Operations

Same as the previous section, let M be a module over a commutative ring R, equipped with a quadratic form $Q : M \to R$.

We also use m or $m_1, m_2, \ldots$ to denote elements of M, i.e. vectors, and x, y, z to denote elements of $C\ell(Q)$.

Definition 2.2.1 (Grade involution)**.** **Grade involution**, intuitively, is negating each basis vector.

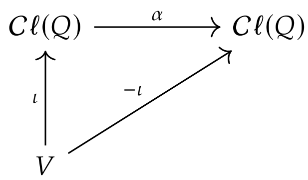
Formally, it's an **algebra homomorphism** $\alpha : C\ell(Q) \to_a C\ell(Q)$, satisfying:

1. $\alpha \circ \alpha = \mathrm{id}$
2. $\alpha(\iota(m)) = -\iota(m)$

for all $m \in M$.

It's called **main involution** α or **main automorphism** in [Jadczyk(2019)], the **canonical automorphism** in [Gallier(2008)].

It's denoted $\hat{m}$ in [Lounesto(2001)], $\alpha(m)$ in [Jadczyk(2019)], m^* in [Chisolm(2012)].



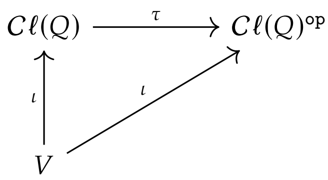
Definition 2.2.2 (Grade reversion)**.** **Grade reversion**, intuitively, is reversing the multiplication order of basis vectors.

Formally, it's an **algebra homomorphism** $\tau : C\ell(Q) \to_a C\ell(Q)^{\mathrm{op}}$, satisfying:

1. $\tau(m_1 m_2) = \tau(m_2)\tau(m_1)$
2. $\tau \circ \tau = \mathrm{id}$
3. $\tau(\iota(m)) = \iota(m)$

It's called **anti-involution** τ in [Jadczyk(2019)], the **canonical anti-automorphism** in [Gallier(2008)], also called **transpose**/**transposition** in some literature, following tensor algebra or matrix.

It's denoted $\tilde{m}$ in [Lounesto(2001)], m^τ in [Jadczyk(2019)] (with variants like m^t or $m^\top$ in other literatures), $m^\dagger$ in [Chisolm(2012)].



Definition 2.2.3 (Clifford conjugate)**. Clifford conjugate** is an **algebra homomorphism** $* : C\ell(Q) \to_a C\ell(Q)$, denoted x^* (or even $x^\dagger$, x^v in some literatures), defined to be:

$$x^* = \text{reverse}(\text{involute}(x)) = \tau(\alpha(x))$$

for all $x \in C\ell(Q)$, satisfying (as a **$*$-ring**):

1. $(x + y)^* = (x)^* + (y)^*$
2. $(xy)^* = (y)^*(x)^*$
3. $* \circ * = \text{id}$
4. $1^* = 1$

and (as a **$*$-algebra**):

$$(rx)^* = r'x^*$$

for all $r \in R$, $x, y \in C\ell(Q)$ where $'$ is the involution of the commutative $*$-ring R.

Note: In our current formalization in Mathlib , the application of the involution on r is ignored, as there appears to be nothing in the literature that advocates doing this.

Clifford conjugate is denoted $\bar{m}$ in [Lounesto(2001)] and most literatures, $m^\ddagger$ in [Chisolm(2012)].

Definition 2.2.4 (Z_2-graded derivations i_f, anti-derivation)**.** We denote by $\text{End}(M)$ the algebra of all **endomorphisms** (linear maps) of M.

For $m \in M$, the linear operator $e_m \in \text{End}(\text{T}(M))$, $\text{T}^p(M) \to \text{T}^{p+1}(M)$ is of left multiplication by m :

$$e_m : x \mapsto e_m(x) = m \otimes x$$

for all $x \in \text{T}(M)$.

Let f be an element of the **dual module** M^*.

The **anti-derivation** $i_f : T(M) \to_{l[T]} (M)$ is a linear map from $T(M)$ to $T(M)$, satisfying:

1. $i_f(1) = 0$
2. $e_m \circ i_f + i_f \circ e_m = f(m) \cdot 1$ for all $m \in M$

The map $f \mapsto i_f$ from M^* to **linear transformations** on $T(M)$ is linear. We have

1. $i_f(m \otimes x) = f(m)x - m \otimes i_f(x)$ for all $m \in M \subset T(M)$, $x \in T(M)$

2. $i_f\,(T^p M) \subset T^{p-1}M$

3. $i_f^2 = 0$

4. $i_f i_g + i_g i_f = 0$, for all $f, g \in M^*$

For $m_1, \ldots, m_p \in M$ we have

$$i_f\left(m_1 \otimes \cdots \otimes m_p\right) = \sum_{i=1}^{p} (-1)^{i-1} f\left(m_i\right) m_1 \otimes \cdots \otimes \check{m}_i \otimes \cdots \otimes m_p$$

where $\check{m}_i$ denotes **deletion** (of m_i from the multiplication).

For a quadratic form Q on M, $\bar{i}_f$ can be defined on the quotient Clifford algebra:

$$\iota \circ i_f = \bar{i}_f \circ \iota,$$

satisfying:

1. $\bar{i}_f(1) = 0$ for $1 \in C\ell(Q)$

2. $\bar{i}_f\,(\iota(m)x) = f(m)x - \iota(m)\bar{i}_f(x)$ for all $m \in M$, $x \in C\ell(Q)$

Let F be a bilinear form on M. Then every $m \in M$ determines a linear form f_m on M defined as $f_m(m') = F(m, m')$.

We will denote by i_m^F the antiderivation i_{f_m}. We have:

1. $i_m^F(1) = 0$,

2. $i_m^F(m' \otimes x) = F(m, m')x - m' \otimes i_m^F(x)$ for all $m' \in M, x \in T(M)$

For $x_1, \ldots, x_n$ in $T(M)$, we have

$$i_m^F\left(x_1 \otimes \cdots \otimes x_n\right) = \sum_{j=1}^{n} (-1)^{j-1} F\left(m, x_j\right) x_1 \otimes \cdots \otimes \check{x}_j \otimes \cdots \otimes x_n$$

As it was in the case with $\bar{i}_f$, we will denote by $\bar{i}_x^F$ the antiderivation $\bar{i}_f$ for $f_m(m') = F(m, m')$:

$$\bar{i}_m^F = \bar{i}_f$$

for $f_m(m') = F(m, m'), (m, m' \in M)$

This is the approach used in [Bourbaki(2007)], and re-introduced in [Jadczyk(2019), Jadczyk(2023)].

$\bar{i}_f$ is denoted ∂_v for $v \in C\ell(Q)_1$ in [Lundholm and Svensson(2009)].

This is closely related to **contraction** (i.e. $\iota(m) \rfloor x = m \rfloor_F x \doteq \bar{i}_m^F(x)$ for $Q = 0$) and **interior product**.

2.3 Structure

2.4 Classification

2.5 Representation

2.6 Spin

3 Geometric Algebra

3.1 Axioms

3.2 Contents

This section would contain what's in Section "The contents of a geometric algebra" in [Chisolm(2012)], e.g. r-blades, r-vectors, before we can discuss anything about the GA operations.

That means we need to first formalize the counter parts in Clifford Algebra, e.g. Lipschitz Group, Spin Group, and Z-filteration in Clifford Algebra.

Jiale Miao's mathlib#16040 (ported to Lean 4 as mathlib4#9111) seems to be a more principled attempt than versors in lean-ga except for the part involving Z-filteration which is still worth porting, possibly with ideas from the prototype here.

We also wish to include some latest results presented in [Ruhe et al.(2023)], with supplements from [Brehmer et al.(2023)], in which some of the results are proven in [Roelfs and De Keninck(2023)].

Definition 3.2.1 (Lipschitz group). TODO

Definition 3.2.2 (Spin group). TODO

Theorem 3.2.3 (The dimension of Clifford algebra)

$\dim C\ell(Q) = 2^n$

where $n = \dim M$.

3.3 Operations and properties

4 Concrete algebras - definition

4.1 CGA

4.2 PGA

4.3 STA

5 Applications

5.1 Geometry

6 Dependency graph

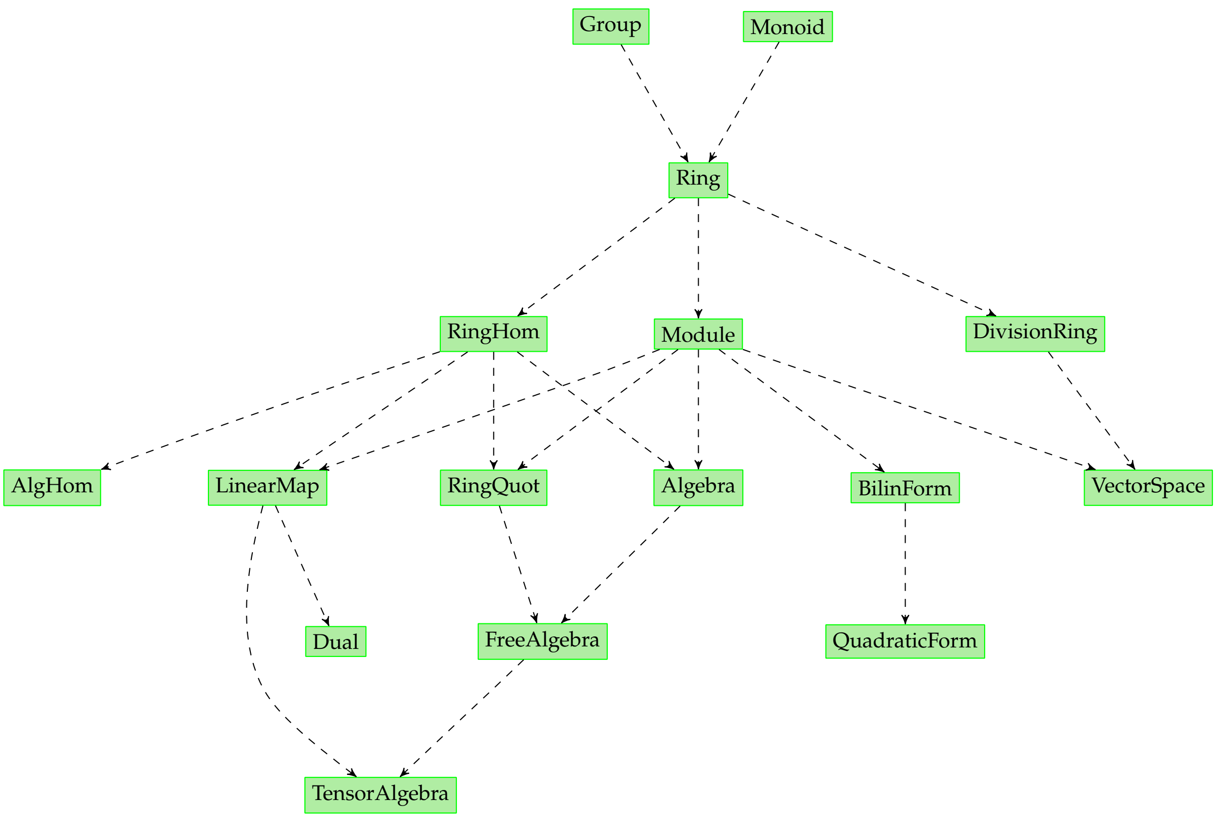


Figure 1: Preliminaries

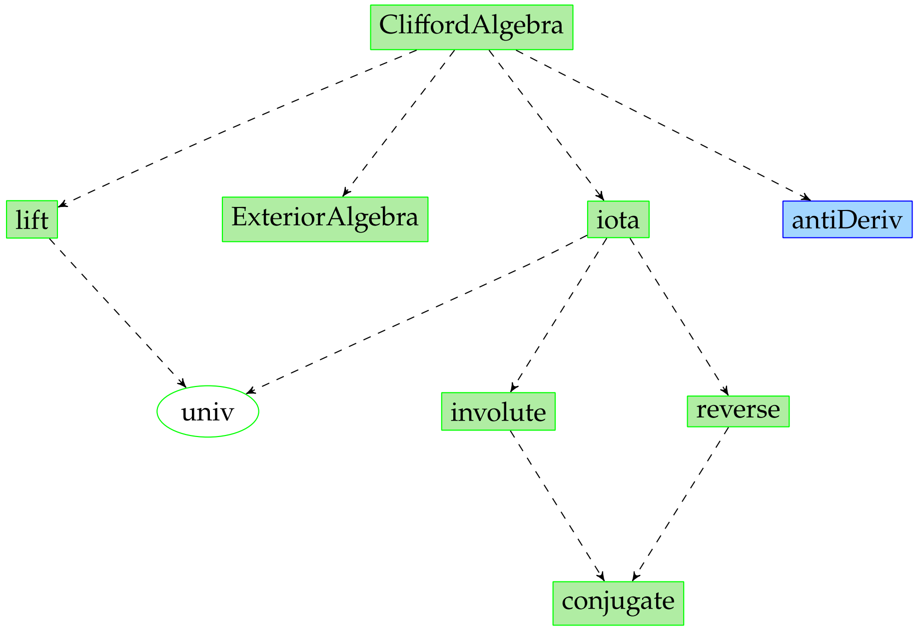


Figure 2: Clifford Algebra

Legends:

- `Boxes`: definitions
- `Ellipses`: theorems and lemmas
- `Blue border`: the *statement* of this result is ready to be formalized; all prerequisites are done
- `Blue background`: the *proof* of this result is ready to be formalized; all prerequisites are done
- `Green border`: the *statement* of this result is formalized
- `Green background`: the *proof* of this result is formalized

References

[Bourbaki(2007)] Nicolas Bourbaki. 2007. *Éléments de mathématique. Algèbre. Chapitre 9* (reprint of the 1959 original ed.). Berlin: Springer. (Cited on pages 9 and 12)

[Brehmer et al.(2023)] Johann Brehmer, Pim De Haan, Sönke Behrends, and Taco Cohen. 2023. Geometric Algebra Transformers. *arXiv preprint arXiv:2305.18415* (2023). (Cited on page 13)

[Carneiro(2019)] Mario Carneiro. 2019. The type theory of Lean. *preparation (https://github.com/digama0/lean-type-theory/releases)* (2019). (Cited on page 3)

[Chen(2016)] Evan Chen. 2016. An infinitely large napkin. (Cited on page 1)

[Chisolm(2012)] Eric Chisolm. 2012. Geometric Algebra. (2012). http://arxiv.org/abs/1205.5935 (Cited on pages 10, 11, and 13)

[de Moura et al.(2015)] Leonardo de Moura, Soonho Kong, Jeremy Avigad, Floris van Doorn, and Jakob von Raumer. 2015. The Lean Theorem Prover (System Description). In *Automated Deduction - CADE-25*, Amy P. Felty and Aart Middeldorp (Eds.). Lecture Notes in Computer Science, Vol. 9195. Springer International Publishing, Cham, 378–388. https://doi.org/10.1007/978-3-319-21401-6_26 (Cited on page 1)

[Gallier(2008)] Jean Gallier. 2008. Clifford algebras, clifford groups, and a generalization of the quaternions. *arXiv preprint arXiv:0805.0311* (2008). (Cited on page 10)

[Garling(2011)] David JH Garling. 2011. *Clifford algebras: an introduction*. Vol. 78. Cambridge University Press. (Cited on page 1)

[Grinberg(2016)] Darij Grinberg. 2016. The Clifford algebra and the Chevalley map - a computational approach. https://www.cip.ifi.lmu.de/~grinberg/algebra/chevalley.pdf (Cited on page 9)

[Hestenes and Sobczyk(1984)] David Hestenes and Garret Sobczyk. 1984. *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics*. Vol. 5. Springer Science & Business Media. https://www.springer.com/gp/book/9789027716736 (Cited on page 1)

[Izhakian et al.(2016)] Zur Izhakian, Manfred Knebusch, and Louis Rowen. 2016. Supertropical quadratic forms I. *Journal of Pure and Applied Algebra* 220, 1 (2016), 61–93. https://doi.org/10.1016/j.jpaa.2015.05.043 (Cited on page 8)

[Jadczyk(2019)] Arkadiusz Jadczyk. 2019. Notes on Clifford Algebras. (2019). (Cited on pages 1, 9, 10, and 12)

[Jadczyk(2023)] Arkadiusz Jadczyk. 2023. On the bundle of Clifford algebras over the space of quadratic forms. *Advances in Applied Clifford Algebras* 33, 2 (2023), 15. (Cited on pages 9 and 12)

[Lounesto(2001)] Pertti Lounesto. 2001. *Clifford Algebras and Spinors*. Vol. 286. Cambridge university press. (Cited on pages 10 and 11)

[Lundholm and Svensson(2009)] Douglas Lundholm and Lars Svensson. 2009. Clifford algebra, geometric algebra, and applications. *arXiv preprint arXiv:0907.5356* (2009). (Cited on page 12)

[Moura and Ullrich(2021)] Leonardo de Moura and Sebastian Ullrich. 2021. The lean 4 theorem prover and programming language. In *Automated Deduction–CADE 28: 28th International Conference on Automated Deduction, Virtual Event, July 12–15, 2021, Proceedings 28*. Springer, 625–635. (Cited on page 1)

[Roelfs and De Keninck(2023)] Martin Roelfs and Steven De Keninck. 2023. Graded symmetry groups: plane and simple. *Advances in Applied Clifford Algebras* 33, 3 (2023), 30. (Cited on page 13)

[Ruhe et al.(2023)] David Ruhe, Johannes Brandstetter, and Patrick Forré. 2023. Clifford group equivariant neural networks. *arXiv preprint arXiv:2305.11141* (2023). (Cited on page 13)

[The mathlib Community(2020)] The mathlib Community. 2020. The Lean Mathematical Library. In *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs* (New Orleans, LA, USA, 2020-01-20) *(CPP 2020)*. Association for Computing Machinery, 367–381. https://doi.org/10.1145/3372885.3373824 (Cited on page 1)

[Ullrich(2023)] Sebastian Andreas Ullrich. 2023. *An Extensible Theorem Proving Frontend*. Ph. D. Dissertation. Dissertation, Karlsruhe, Karlsruher Institut für Technologie (KIT), 2023. (Cited on pages 1 and 3)

[Wieser and Song(2022)] Eric Wieser and Utensil Song. 2022. Formalizing geometric algebra in lean. *Advances in Applied Clifford Algebras* 32, 3 (2022), 28. (Cited on page 9)