Compendium on Spin groups

Utensil Song

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1 Motivation

This survey is built on my notes in the process of figuring out Eric Wieser's MathOverflow question Definition of a spin group for our PR to Lean 4's Mathlib about Spin groups.

This is my first Forester experiment, also a spiritual successor to my writeup The Many Faces of Geometric Algebra.

2 Our definition

Definition 2.0.1 (Clifford algebra [Miao et al.(2024)])

Let M be a module over a commutative ring R, equipped with a quadratic form $Q : M \to R$.

A Clifford algebra over Q is

$$\mathcal{C}\ell(Q) \equiv T(M)/I_Q \tag{2.0.2}$$

where $T(M)$ is the tensor algebra of M, I_Q is the two-sided ideal generated from the set

$$\{m \otimes m - Q(m) \mid m \in M\}. \tag{2.0.3}$$

We denote the canonical linear map $M \to \mathcal{C}\ell(Q)$ as ι_Q.

Definition 2.0.4 (Lipschitz-Clifford group [Miao et al.(2024)])

The Lipschitz-Clifford group is defined as the subgroup closure of all the invertible elements in the form of $\iota_Q(m)$,

$$\Gamma \equiv \{x_1 \ldots x_k \in \mathcal{C}\ell^{\times}(Q) \mid x_i \in V\} \tag{2.0.5}$$

where

$$\mathcal{C}\ell^{\times}(Q) \equiv \left\{x \in \mathcal{C}\ell(Q) \mid \exists x^{-1} \in \mathcal{C}\ell(Q), x^{-1}x = xx^{-1} = 1\right\} \tag{2.0.6}$$

is the group of units (i.e. invertible elements) of $\mathcal{C}\ell(Q)$,

$$V \equiv \left\{\iota_Q(m) \in \mathcal{C}\ell(Q) \mid m \in M\right\} \tag{2.0.7}$$

is the subset V of $\mathcal{C}\ell(Q)$ in the form of $\iota_Q(m)$.

Definition 2.0.8 (Pin group [Miao et al.(2024)])
The Pin group is defined as

$$\mathrm{Pin}(Q) \equiv \Gamma \sqcap \mathrm{U}(\mathcal{C}\ell(Q)) \tag{2.0.9}$$

where $\sqcap$ is the infimum (or greatest lower bound, or meet), and the infimum of two submonoids is just their intersection $\cap$,

$$\mathrm{U}(S) \equiv \{x \in S \mid x^* * x = x * x^* = 1\} \tag{2.0.10}$$

are the unitary elements of the Clifford algebra as a $*$-monid, and we have defined the star operation of Clifford algebra as Clifford conjugate [wieser2022formalizing], denoted $\bar{x}$.

This definition is equivalent to the following:

$$\mathrm{Pin}(Q) \equiv \{x \in \Gamma \mid \mathrm{N}(x) = 1\} \tag{2.0.11}$$

where

$$\mathrm{N}(x) \equiv x\bar{x}. \tag{2.0.12}$$

Definition 2.0.13 (Spin group [Miao et al.(2024)])
The Spin group is defined as

$$\mathrm{Spin}(Q) \equiv \mathrm{Pin}(Q) \sqcap \mathcal{C}\ell^{+}(Q) \tag{2.0.14}$$

where $\mathcal{C}\ell^{+}(Q)$ is the even subalgebra of the Clifford algebra.

3 Appendix: Many faces of Spin group

Definitions coming from different sources are simply quoted here with minimal modifications, to include immediate prerequisites, and omit some discussions or theorems.

They are not classified, ordered, or pruned by similarity.

Definition 3.0.1 (Spin group [Lawson and Michelsohn(2016)])
Let V be a vector space over the commutative field k and suppose q is a quadratic form on V.

We now consider the multiplicative group of units in the Clifford algebra $C\ell(V,q)$ associated to V, which is defined to be the subset

$$C\ell^{\times}(V,q) \equiv \left\{\varphi \in C\ell(V,q) : \exists\varphi^{-1} \text{ with } \varphi^{-1}\varphi = \varphi\varphi^{-1} = 1\right\} \tag{3.0.2}$$

This group contains all elements $v \in V$ with $q(v) \neq 0$.

The group of units always acts naturally as automorphisms of the algebra. That is, there is a homomorphism

$$\mathrm{Ad} : \mathrm{C}\ell^{\times}(V,q) \longrightarrow \mathrm{Aut}(\mathrm{C}\ell(V,q)) \tag{3.0.3}$$

called the adjoint representation, which is given by

$$\mathrm{Ad}_{\varphi}(x) \equiv \varphi \times \varphi^{-1} \tag{3.0.4}$$

The Pin group of (V,q) is the subgroup $\mathrm{Pin}(V,q)$ of $\mathrm{P}(V,q)$ generated by the elements $v \in V$ with $q(v) = \pm 1$.

The associated Spin group of (V,q) is defined by

$$\mathrm{Spin}(V,q) = \mathrm{Pin}(V,q) \cap C\ell^{0}(V,q). \tag{3.0.5}$$

Definition 3.0.6 (Spin group [Wikipedia(2024)])

The pin group $\mathrm{Pin}_V(K)$ is the subgroup of the Lipschitz group Γ of elements of spinor norm 1, and similarly the spin group $\mathrm{Spin}_V(K)$ is the subgroup of elements of Dickson invariant 0 in $\mathrm{Pin}_V(K)$.

Definition 3.0.7 (Spin group [Li(2008)])

A versor refers to a Clifford monomial composed of invertible vectors. It is called a rotor, or spinor, if the number of vectors is even. It is called a unit versor if its magnitude is 1.

All versors in $\mathcal{CL}(\mathcal{V}^n)$ form a group under the geometric product, called the versor group, also known as the Clifford group, or Lipschitz group. All rotors form a subgroup, called the rotor group. All unit versors form a subgroup, called the pin group, and all unit rotors form a subgroup, called the spin group, denoted by $\mathrm{Spin}(\mathcal{V}^n)$.

Definition 3.0.8 (Spin group [Sommer(2013)])

The Clifford group $\Gamma_{\mathrm{p,q}}$ of a Clifford algebra $\mathcal{C}_{p,q}$ is defined as

$$\Gamma_{\mathrm{p,q}} := \left\{ s \in \mathcal{C}_{p,q} \mid \forall x \in \mathbb{R}_{p,q} : s x \hat{s}^{-1} \in \mathbb{R}_{p,q} \right\}. \tag{3.0.9}$$

From that definition we get immediately

$$\Gamma_{\mathrm{p,q}} \times \mathbb{R}_{p,q} \to \mathbb{R}_{p,q}; \quad (s, x) \mapsto s x \hat{s}^{-1} \tag{3.0.10}$$

as the operation of the Clifford group $\Gamma_{\mathrm{p,q}}$ on $\mathbb{R}_{p,q}$.

$\Gamma_{\mathrm{p,q}}$ is a multiple cover of the orthogonal group $O(p, q)$. However, it is still unnecessarily large. Therefore, we first reduce $\Gamma_{\mathrm{p,q}}$ to a two-fold cover of $O(p, q)$ by defining the so-called Pin group

$$\mathrm{Pin(p, q)} := \left\{ s \in \Gamma_{\mathrm{p,q}} \mid s\tilde{s} = \pm 1 \right\}. \tag{3.0.11}$$

The even elements of $\mathrm{Pin}(p, q)$ form the spin group

$$\mathrm{Spin(p, q)} := \mathrm{Pin(p, q)} \cap \mathcal{C}^{+}_{p,q} \tag{3.0.12}$$

which is a double cover of the special orthogonal group $SO(p, q)$. Finally, those elements of $\mathrm{Spin(p, q)}$ with Clifford norm equal 1 form a further subgroup

$$\mathrm{Spin}_{+}\mathrm{(p, q)} := \{ s \in \mathrm{Spin(p, q)} \mid s\tilde{s} = 1 \} \tag{3.0.13}$$

that covers $\mathrm{SO}_{+}(p, q)$ twice. Thereby, $\mathrm{SO}_{+}(p, q)$ is the connected component of the identity of $O(p, q)$.

Definition 3.0.14 (Spin group [Perwass et al.(2009)])
A versor is a multivector that can be expressed as the geometric product of a number of non-null 1-vectors. That is, a versor $\boldsymbol{V}$ can be written as $\boldsymbol{V} = \prod_{i=1}^{k} \boldsymbol{n}_i$, where $\{\boldsymbol{n}_1, \ldots, \boldsymbol{n}_k\} \subset \mathbb{G}^{\varnothing 1}_{p,q}$ with $k \in \mathbb{N}^+$, is a set of not necessarily linearly independent vectors.

The subset of versors of $\mathbb{G}_{p,q}$ together with the geometric product, forms a group, the Clifford group, denoted by $\mathfrak{G}_{p,q}$.

A versor $\boldsymbol{V} \in \mathfrak{G}_{p,q}$ is called unitary if $\boldsymbol{V}^{-1} = \tilde{\boldsymbol{V}}$, i.e. $\boldsymbol{V}\widetilde{\boldsymbol{V}} = +1$.

The set of unitary versors of $\mathfrak{G}_{p,q}$ forms a subgroup $\mathfrak{P}_{p,q}$ of the Clifford group $\mathfrak{G}_{p,q}$, called the pin group.

A versor $\boldsymbol{V} \in \mathfrak{G}_{p,q}$ is called a spinor if it is unitary ($\boldsymbol{V}\tilde{\boldsymbol{V}} = 1$) and can be expressed as the geometric product of an even number of 1-vectors. This implies that a spinor is a linear combination of blades of even grade.

The set of spinors of $\mathfrak{G}_{p,q}$ forms a subgroup of the pin group $\mathfrak{P}_{p,q}$, called the spin group, which is denoted by $\mathfrak{S}_{p,q}$.

Definition 3.0.15 (Spin group [Jadczyk(2019)])
We define the Clifford group $\Gamma = \Gamma(q)$ to be the group of all invertible elements $u \in \mathrm{Cl}(q)$ which have the property that uyu^{-1} is in M whenever y is in M. We define $\Gamma(q)^{\pm}$as the intersection of $\Gamma(q)$ and $\mathrm{Cl}(q)_{\pm}$.

For every element $u \in \Gamma(q)$ we define the spinor norm $N(u)$ by the formula

$$N(u) = \tau(u)u, \tag{3.0.16}$$

where τ is the main involution of the Clifford algebra $\mathrm{Cl}(q)$.

The following groups are called spin groups:

$$\begin{aligned} \mathrm{Pin}(q) &:= \left\{s \in \Gamma(q)^+ \cup \Gamma(q)^- : N(s) = \pm 1\right\} \\ \mathrm{Spin}(q) &:= \left\{s \in \Gamma(q)^+ : N(s) = \pm 1\right\} \\ \mathrm{Spin}^+(q) &:= \left\{s \in \Gamma(q)^+ : N(s) = +1\right\}. \end{aligned} \tag{3.0.17}$$

Definition 3.0.18 (Spin group [Garling(2011)])
Suppose that (E, q) is a regular quadratic space. We consider the action of $\mathcal{G}(E, q)$ on $\mathcal{A}(E, q)$ by adjoint conjugation. We set

$$Ad'_g(a) = gag^{-1}, \tag{3.0.19}$$

for $g \in \mathcal{G}(E, q)$ and $a \in \mathcal{A}(E, q)$.

We restrict attention to those elements of $\mathcal{G}(E,q)$ which stabilize E. The Clifford group $\Gamma = \Gamma(E,q)$ is defined as

$$\left\{ g \in \mathcal{G}(E,q) : Ad'_g(x) \in E \text{ for } x \in E \right\}. \tag{3.0.20}$$

If $g \in \Gamma(E,q)$, we set $\alpha(g)(x) = Ad'_g(x)$. Then $\alpha(g) \in GL(E)$, and α is a homomorphism of Γ into $GL(E).\alpha$ is called the graded vector representation of Γ.

It is customary to scale the elements of $\Gamma(E,q)$; we set

$$\begin{aligned} \mathrm{Pin}(E,q) &= \{g \in \Gamma(E,q) : \Delta(g) = \pm 1\}, \\ \Gamma_1(E,q) &= \{g \in \Gamma(E,q) : \Delta(g) = 1\}. \end{aligned} \tag{3.0.21}$$

If (E,q) is a Euclidean space, then $\mathrm{Pin}(E,q) = \Gamma_1(E,q)$; otherwise, $\Gamma_1(E,q)$ is a subgroup of $\mathrm{Pin}(E,q)$ of index 2. We have a short exact sequence

$$1 \longrightarrow D_2 \xrightarrow{\subseteq} \mathrm{Pin}(E,q) \xrightarrow{\alpha} O(E,q) \longrightarrow 1; \tag{3.0.22}$$

$\mathrm{Pin}(E,q)$ is a double cover of $O(E,q)$.

In fact there is more interest in the subgroup $\mathrm{Spin}(E,q)$ of $\mathrm{Pin}(E,q)$ consisting of products of an even number of unit vectors in E. Thus $\mathrm{Spin}(E,q) = \mathrm{Pin}(E,q) \cap \mathcal{A}^+(E,q)$ and

$$\mathrm{Spin}(E,q) = \left\{ g \in \mathcal{A}^+(E,q) : gE = Eg \text{ and } \Delta(g) = \pm 1 \right\}. \tag{3.0.23}$$

If x, y are unit vectors in E then $\alpha(xy) = \alpha(x)\alpha(y) \in SO(E,q)$, so that $\alpha(\mathrm{Spin}(E,q)) \subseteq SO(E,q)$. Conversely, every element of $SO(E,q)$ is the product of an even number of simple reflections, and so $SO(E,q) \subseteq \alpha\left(\mathrm{Spin}(E,q)\right)$. Thus $\alpha\left(\mathrm{Spin}(E,q)\right) = SO(E,q)$, and we have a short exact sequence.

$$1 \longrightarrow D_2 \xrightarrow{\subseteq} \mathrm{Spin}(E,q) \xrightarrow{\alpha} SO(E,q) \longrightarrow 1; \tag{3.0.24}$$

$\mathrm{Spin}(E,q)$ is a double cover of $SO(E,q)$.

Note also that if $a \in \mathrm{Spin}(E,q)$ and $x \in E$ then $\alpha(a)(x) = axa^{-1}$; conjugation and adjoint conjugation by elements of $\mathrm{Spin}(E,q)$ are the same.

Definition 3.0.25 (Spin group [Meinrenken(2009)])
Recall that $\Pi : \mathrm{Cl}(V) \to \mathrm{Cl}(V), x \mapsto (-1)^{|x|}x$ denotes the parity automorphism of the Clifford algebra. Let $\mathrm{Cl}(V)^\times$be the group of invertible elements in $\mathrm{Cl}(V)$.

The Clifford group $\Gamma(V)$ is the subgroup of $\mathrm{Cl}(V)^\times$, consisting of all $x \in \mathrm{Cl}(V)^\times$ such that $A_x(v) := \Pi(x)vx^{-1} \in V$ for all $v \in V \subset \mathrm{Cl}(V)$.

Hence, by definition the Clifford group comes with a natural representation, $\Gamma(V) \to \mathrm{GL}(V)$, $x \mapsto A_x$. Let $S\Gamma(V) = \Gamma(V) \cap \mathrm{Cl}^{\overline{0}}(V)^{\times}$denote the special Clifford group.

The canonical representation of the Clifford group takes values in $\mathrm{O}(V)$, and defines an exact sequence,

$$1 \longrightarrow \mathbb{K}^{\times} \longrightarrow \Gamma(V) \longrightarrow \mathrm{O}(V) \longrightarrow 1. \tag{3.0.26}$$

It restricts to a similar exact sequence for the special Clifford group,

$$1 \longrightarrow \mathbb{K}^{\times} \longrightarrow S\Gamma(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1. \tag{3.0.27}$$

The elements of $\Gamma(V)$ are all products $x = v_1 \cdots v_k$ where $v_1, \dots, v_k \in V$ are non-isotropic. $S\Gamma(V)$ consists of similar products, with k even. The corresponding element A_x is a product of reflections:

$$A_{v_1 \cdots v_k} = R_{v_1} \cdots R_{v_k}. \tag{3.0.28}$$

Suppose $\mathbb{K} = \mathbb{R}$. The Pin group $\mathrm{Pin}(V)$ is the preimage of $\{1, -1\}$ under the norm homomorphism $N : \Gamma(V) \to \mathbb{K}^{\times}$. Its intersection with $S\Gamma(V)$ is called the Spin group, and is denoted $\mathrm{Spin}(V)$.

Since $N(\lambda) = \lambda^2$ for $\lambda \in \mathbb{K}^{\times}$, the only scalars in $\mathrm{Pin}(V)$ are ± 1. Hence, the exact sequence for the Clifford group restricts to an exact sequence,

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}(V) \longrightarrow \mathrm{O}(V) \longrightarrow 1, \tag{3.0.29}$$

so that $\mathrm{Pin}(V)$ is a double cover of $\mathrm{O}(V)$. Similarly,

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1, \tag{3.0.30}$$

defines a double cover of $\mathrm{SO}(V)$. Elements in $\mathrm{Pin}(V)$ are products $x = v_1 \cdots v_k$ with $B(v_i, v_i) = \pm 1$. The group $\mathrm{Spin}(V)$ consists of similar products, with k even.

Definition 3.0.31 (Spin group [Weber(2013)])
The "group of units" in $Cl_{p,q}$, denoted $Cl^{\times}_{p,q}$, is the group of all invertible elements.

An important subgroup of $Cl^{\times}(V, q)$ is the group $P(V, q)$ generated by elements $v \in V$ with $q(v) \neq 0$. Quotienting out by constants, we obtain the Pin group. Specifically, $\mathrm{Pin}(V, q)$ (or $\mathrm{Pin}_{p,q}$) is the group generated by elements $v \in V$ with $q(v) = \pm 1$. Further define the spin groups to be

$$\mathrm{Spin}(V, q) = \mathrm{Pin}(V, q) \cap Cl^0(V, q). \tag{3.0.32}$$

Definition 3.0.33 (Spin group [Bär(2018)])

We define the Pin group $\mathrm{Pin}(\boldsymbol{n})$ by

$$\mathrm{Pin}(n) := \left\{ v_1 \cdot \ldots \cdot v_m \in \mathrm{Cl}_n \mid v_j \in S^{n-1} \subset \mathbb{R}^n, m \in \mathbb{N}_0 \right\} \tag{3.0.34}$$

We define the Spin group $\mathrm{Spin}(\boldsymbol{n})$ by

$$\begin{aligned} \mathrm{Spin}(n) &:= \mathrm{Pin}(n) \cap \mathrm{Cl}_n^0 \\ &= \left\{ v_1 \cdot \ldots \cdot v_m \in \mathrm{Cl}_n \mid v_j \in S^{n-1}, m \in 2\mathbb{N}_0 \right\} \end{aligned} \tag{3.0.35}$$

Definition 3.0.36 (Spin group [Woit(2012)])

There are several equivalent possible ways to go about defining the $\mathrm{Spin}(n)$ groups as groups of invertible elements in the Clifford algebra.

1. One can define $\mathrm{Spin}(n)$ in terms of invertible elements $\tilde{g}$ of $C_{\text{even}}(n)$ that leave the space $V = \mathbf{R}^n$ invariant under conjugation

$$\tilde{g} V \tilde{g}^{-1} \subset V \tag{3.0.37}$$

2. One can show that, for $v, w \in V$,

$$v \rightarrow v - 2\frac{Q(v,w)}{Q(w,w)} w = -wvw/Q(w,w) = wvw^{-1} \tag{3.0.38}$$

is reflection in the hyperplane perpendicular to w. Then $\mathrm{Pin}(n)$ is defined as the group generated by such reflections with $||w||^2 = 1$. $\mathrm{Spin}(n)$ is the subgroup of $\mathrm{Pin}(n)$ of even elements. Any rotation can be implemented as an even number of reflections (Cartan-Dieudonné) theorem.

3. One can define the Lie algebra of $\mathrm{Spin}(n)$ in terms of quadratic elements of the Clifford algebra.

Definition 3.0.39 (Spin group [Liu(2016)])

The space of quadratic vectors in Cl is the Lie algebra of $\mathrm{SO}(n)$. The corresponding Lie group, called the Spin group $\mathrm{Spin}(Q)$, is the set of invertible elements $x \in \mathrm{Cl}$ that preserve V under $v \mapsto xvx^{-1}$. Clearly this map is in $\mathrm{SO}(V,Q)$ since it preserves the quadratic form Q, and is a two-fold cover with kernel ± 1.

Definition 3.0.40 (Spin group [Figueroa-O'Farrill(2010)])

The Pin group $\mathrm{Pin}\,(V)$ of (V,Q) is the subgroup of (the group of units of) $C\ell(V)$ generated by $v \in \mathrm{V}$ with $\mathrm{Q}(v) = \pm 1$. In other words, every element of $\mathrm{Pin(V)}$ is of the form $u_1 \cdots u_r$ where $u_i \in \mathrm{V}$ and $\mathrm{Q}\,(u_i) = \pm 1$. We will write $\mathrm{Pin}(s,t)$ for $\mathrm{Pin}\left(\mathbb{R}^{s,t}\right)$ and $\mathrm{Pin}(n)$ for $\mathrm{Pin}(n,0)$.

The spin group of (V,Q) is the intersection

$$\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap C\ell(V)_0. \tag{3.0.41}$$

Equivalently, it consists of elements $u_1 \cdots u_{2p}$, where $u_i \in \mathrm{V}$ and $\mathrm{Q}(u_i) = \pm 1$. We will write $\mathrm{Spin}(s,t)$ for $\mathrm{Spin}\left(\mathbb{R}^{s,t}\right)$ and $\mathrm{Spin}(n)$ for $\mathrm{Spin}(n,0)$.

Definition 3.0.42 (Spin group [Lundholm and Svensson(2009)])
We identify the following groups embedded in $\mathcal{G}$:

$$\begin{aligned}
\mathcal{G}^{\times} &:= \{x \in \mathcal{G} : \exists y \in \mathcal{G} : xy = yx = 1\} && \text{the group of all invertible elements}\\
\tilde{\Gamma} &:= \left\{x \in \mathcal{G}^{\times} : x^{\star} V x^{-1} \subseteq V\right\} && \text{the Lipschitz group}\\
\Gamma &:= \{v_1 v_2 \ldots v_k \in \mathcal{G} : v_i \in V^{\times}\} && \text{the versor group}\\
\text{Pin} &:= \left\{x \in \Gamma : x x^{\dagger} \in \{-1,1\}\right\} && \text{the group of unit versors}\\
\text{Spin} &:= \mathrm{Pin} \cap \mathcal{G}^{+} && \text{the group of even unit versors}\\
\text{Spin}^{+} &:= \left\{x \in \mathrm{Spin} : x x^{\dagger} = 1\right\} && \text{the rotor group}
\end{aligned} \tag{3.0.43}$$

In the above, $V^{\times} := \left\{v \in V : v^2 \neq 0\right\}$ denotes the set of all invertible vectors.

Definition 3.0.44 (Spin group [Renaud(2020)])
The Clifford group $\Gamma(p,q)$ is the (multiplicative) group generated by invertible 1-vectors in $R^{p,q}$.

The Pin group $\mathrm{Pin}(p,q)$.

$$\mathrm{Pin}(p,q) = \{g \in \Gamma(p,q) : g\widetilde{g} = \pm 1\}. \tag{3.0.45}$$

So if $g \in \mathrm{Pin}(p,q)$, $\widetilde{g}$ is a scalar multiple of g^{-1}. This is not true for arbitrary elements of $\Gamma(p,q)$.

The Spin group $\mathrm{Spin}(p,q)$. This is the subgroup of $\mathrm{Pin}(p,q)$ consisting of even elements only, i.e.

$$\mathrm{Spin}(p,q) = \mathrm{Pin}(p,q) \cap Cl^{+}(p,q). \tag{3.0.46}$$

The Spin group $\mathrm{Spin}(p,q)$ has the further subgroup

$$\mathrm{Spin}^{\dagger}(p,q) = \{g \in \mathrm{Spin}(p,q) : g\widetilde{g} = +1\}. \tag{3.0.47}$$

$\mathrm{Pin}(p,q)$, $\mathrm{Spin}(p,q)$ and $\mathrm{Spin}^{\dagger}(p,q)$ are respectively the two-fold covering groups of $O(p,q)$, $SO(p,q)$ and $SO^{\dagger}(p,q)$ (where $SO^{\dagger}(p,q)$ is the connected component of $SO(p,q)$).

Definition 3.0.48 (Spin group [Dutailly(2018)])
The Spin group $\mathrm{Spin}(F,\rho)$ of $Cl(F,\rho)$ is the subset of $Cl(F,\rho)$ whose elements can be written as the product $g = u_1 \cdot \ldots \cdot u_{2p}$ of an even number of vectors of F of norm $\langle u_k, u_k \rangle = 1$.

As a consequence : $\langle g, g\rangle = 1, g^t \cdot g = 1$ and $\mathrm{Spin}(F, \rho) \subset O(Cl)$.

The scalars ± 1 belong to the Spin group. The identity is $+1$. $\mathrm{Spin}(F, \rho)$ is a connected Lie group.

Definition 3.0.49 (Spin group [Hitzer(2012)])

A versor refers to a Clifford monomial (product expression) composed of invertible vectors. It is called a rotor, or spinor, if the number of vectors is even. It is called a unit versor if its magnitude is 1.

Every versor $A = a_1 \dots a_r, \quad a_1, \dots, a_r \in \mathbb{R}^2, r \in \mathbb{N}$ has an inverse

$$A^{-1} = a_r^{-1} \dots a_1^{-1} = a_r \dots a_1 / \left(a_1^2 \dots a_r^2\right), \tag{3.0.50}$$

such that

$$AA^{-1} = A^{-1}A = 1. \tag{3.0.51}$$

This makes the set of all versors in $Cl(2, 0)$ a group, the so called Lipschitz group with symbol $\Gamma(2, 0)$, also called Clifford group or versor group. Versor transformations apply via outermorphisms to all elements of a Clifford algebra. It is the group of all reflections and rotations of $\mathbb{R}^2$.

The normalized subgroup of versors is called pin group

$$\mathrm{Pin}(2, 0) = \{A \in \Gamma(2, 0) \mid A\widetilde{A} = \pm 1\}. \tag{3.0.52}$$

In the case of $Cl(2, 0)$ we have

$$\begin{aligned}&\mathrm{Pin}(2, 0)\\ &= \left\{a \in \mathbb{R}^2 \mid a^2 = 1\right\} \cup \left\{A \mid A = \cos\varphi + e_{12}\sin\varphi, \varphi \in \mathbb{R}\right\}.\end{aligned} \tag{3.0.53}$$

The pin group has an even subgroup, called spin group

$$\mathrm{Spin}(2, 0) = \mathrm{Pin}(2, 0) \cap Cl^+(2, 0). \tag{3.0.54}$$

The spin group has in general a spin plus subgroup

$$\mathrm{Spin}_+(2, 0) = \{A \in \mathrm{Spin}(2, 0) \mid A\widetilde{A} = +1\}. \tag{3.0.55}$$

Definition 3.0.56 (Spin group [Hahn(2004)])

We continue to let F be a field of characteristic not 2 and M a quadratic space over F.

Recall that $\gamma : M \to C(M)$ is injective and that there is a unique involution - on $C(M)$ taking γx to γx for all x. Consider M to be a subset of $C(M)$ via γ, and define the group $\mathrm{Spin}(M)$ by

$$\mathrm{Spin}(M) = \left\{c \in C_0(M)^\times \mid cMc^{-1} = M, c\bar{c} = 1_C\right\}, \tag{3.0.57}$$

where $C_0(M)^\times$is the group of invertible elements of the ring $C_0(M)$. The isometries from M onto M constitute the orthogonal group $O(M)$ and $SO(M)$ is the subgroup of elements of determinant 1. For c in $\mathrm{Spin}(M)$, define

$$\pi c : M \to M \tag{3.0.58}$$

by $\pi c(x) = cxc^{-1}$. This provides a homomorphism

$$\pi : \mathrm{Spin}(M) \to SO(M). \tag{3.0.59}$$

By a theorem of Cartan and Dieudonné, any element σ in $O(M)$ is a product $\sigma = \tau_{y_1} \cdots \tau_{y_k}$ of hyperplane reflections τ_{y_i}. The assignment $\Theta(\sigma) = q\left(y_1\right) \cdots q\left(y_k\right)(F^\times)^2$ defines the spinor norm homomorphism

$$\Theta : SO(M) \to F^\times/(F^\times)^2. \tag{3.0.60}$$

Definition 3.0.61 (Spin group [Porteous(1995)])
Let g be an invertible element of a universal Clifford algebra A such that, for each $x \in X$, $gx\widehat{g}^{-1} \in X$. Then the map

$$\rho_{X,g} : x \mapsto gx\widehat{g}^{-1} \tag{3.0.62}$$

is an orthogonal automorphism of X.

The element g will be said to induce or represent the orthogonal transformation $\rho_{X,g}$ and the set of all such elements g will be denoted by $\Gamma(X)$ or simply by Γ.

The subset Γ is a subgroup of A.

The group Γ is called the Clifford group (or Lipschitz group) for X in the Clifford algebra A. Since the universal algebra A is uniquely defined up to isomorphism, Γ is also uniquely defined up to isomorphism.

An element g of $\Gamma(X)$ represents a rotation of X if and only if g is the product of an even number of elements of X. The set of such elements will be denoted by $\Gamma^0 = \Gamma^0(X)$. An element g of Γ represents an anti-rotation of X if and only if g is the product of an odd number of elements of X. The set of such elements will be denoted by $\Gamma^1 = \Gamma^1(X)$. Clearly, $\Gamma^0 = \Gamma \cap A^0$ is a subgroup of Γ, while $\Gamma^1 = \Gamma \cap A^1$.

The Clifford group $\Gamma(X)$ of a quadratic space X is larger than is necessary if our interest is in representing orthogonal transformations of X. Use of a quadratic norm N on the Clifford algebra A leads to the definition of subgroups of Γ that are less redundant for this purpose. This quadratic norm $N : A \longrightarrow A$ is defined by the formula

$$N(a) = a^{-}a, \text{ for any } a \in A, \tag{3.0.63}$$

For X and $\Gamma = \Gamma(X)$ as above we now define

$$\operatorname{Pin} X = \{g \in \Gamma : |N(g)| = 1\} \text{ and } \operatorname{Spin} X = \left\{g \in \Gamma^0 : |N(g)| = 1\right\}. \tag{3.0.64}$$

Definition 3.0.65 (Spin group [Rosén(2019)])

Let V be an inner product space. We denote by ΔV the standard Clifford algebra $(\wedge V, +, \Delta)$ defined by the Clifford product Δ on the space of multivectors in V.

Let V be an inner product space. The Clifford cone of V is the multiplicative group $\widehat{\triangle} V \subset \triangle V$ generated by nonsingular vectors, that is, vectors v such that $\langle v\rangle^2 \neq 0$. More precisely, $q \in \widehat{\triangle} V$ if there are finitely many nonsingular vectors $v_1, \ldots, v_k \in V$ such that

$$q = v_1 \Delta \cdots \Delta v_k. \tag{3.0.66}$$

Let $w \in \triangle V$. Then $w \in \widehat{\triangle} V$ if and only if w is invertible and $\widehat{w} v w^{-1} \in V$ for all $v \in V$.

In this case w can be written as a product of at most $\dim V$ nonsingular vectors, and $\bar{w} w = w \bar{w} \in \mathbf{R} \backslash \{0\}$.

Let V be an inner product space. Define the orthogonal, special orthogonal, pin, and spin groups

$$\begin{aligned}
\mathrm{O}(V) &:= \{ \text{ isometries } T : V \to V\} \subset \mathcal{L}(V), \\
\mathrm{SO}(V) &:= \{T \in \mathrm{O}(V); \det T = +1\} \subset \mathcal{L}(V), \\
\mathrm{Pin}(V) &:= \left\{q \in \widehat{\triangle} V; \langle q\rangle^2 = \pm 1\right\} \subset \triangle V, \\
\mathrm{Spin}(V) &:= \left\{q \in \mathrm{Pin}(V); q \in \triangle^{\mathrm{ev}} V\right\} \subset \triangle^{\mathrm{ev}} V.
\end{aligned} \tag{3.0.67}$$

We call $T \in \mathrm{SO}(V)$ a rotation and we call $q \in \mathrm{Spin}(V)$ a rotor.

Definition 3.0.68 (Spin group [Ruhe et al.(2024)])

Motivation E. 39 (The problem of generalizing the definition of the Spin group). For a positive definite quadratic form $\mathfrak{q}$ on the real vector space $V = \mathbb{R}^n$ with $n \geq 3$ the Spin group $\mathrm{Spin}(n)$ is defined via the kernel of the Spinor norm (=extended quadratic form on $\mathrm{Cl}(V, \mathfrak{q})$) restricted to the special Clifford group

$\Gamma^{[0]}(V,\mathfrak{q})$:

$$\mathrm{Spin}(n) := \ker\left(\bar{\mathfrak{q}} : \Gamma^{[0]}(V,\mathfrak{q}) \to \mathbb{R}^{\times}\right) = \left\{w \in \Gamma^{[0]}(V,\mathfrak{q}) \mid \bar{\mathfrak{q}}(w) = 1\right\} = \bar{\mathfrak{q}}|^{-1}_{\Gamma^{[0]}(V,\mathfrak{q})}(1). \tag{3.0.69}$$

$\mathrm{Spin}(n)$ is thus a normal subgroup of the special Clifford group $\Gamma^{[0]}(V,\mathfrak{q})$, and, as it turns out, a double cover of the special orthogonal group $\mathrm{SO}(n)$ via the twisted conjugation ρ. The latter can be summarized by the short exact sequence:

$$1 \longrightarrow \{\pm 1\} \xrightarrow{\text{incl}} \mathrm{Spin}(n) \xrightarrow{\rho} \mathrm{SO}(n) \longrightarrow 1. \tag{3.0.70}$$

We intend to generalize this in several directions: 1. from Spin to Pin group, 2. from $\mathbb{R}^n$ to vector spaces V over general fields $\mathbb{F}$ with $\mathrm{ch}(\mathbb{F}) \neq 2$, 3. from non-degenerate to degenerate quadratic forms $\mathfrak{q}$, 4. from positive (semi-)definite to non-definite quadratic forms $\mathfrak{q}$. This comes with several challenges and ambiguities.

Definition E. 40 (The real Pin group and the real Spin group). Let V be a finite dimensional $\mathbb{R}$-vector space $V, \dim V = n < \infty$, and $\mathfrak{q}a$ (possibly degenerate) quadratic form on V. We define the (real) Pin group and (real) Spin group, resp., of $(V,\mathfrak{q})$ as the following subquotients of the Clifford group. $\Gamma(V,\mathfrak{q})$ and its even parity part $\Gamma^{[0]}(V,\mathfrak{q})$, resp.:

$$\begin{aligned}
\mathrm{Pin}(V,\mathfrak{q}) &:= \{x \in \Gamma(V,\mathfrak{q}) \mid \bar{\mathfrak{q}}(x) \in \{\pm 1\}\} / \overset{[*]}{\bigwedge}(\mathcal{R}) \\
\mathrm{Spin}_{\infty}(V,\mathfrak{q}) &:= \left\{x \in \Gamma^{[0]}(V,\mathfrak{q}) \mid \bar{\mathfrak{q}}(x) \in \{\pm 1\}\right\} / \overset{[*]}{\bigwedge}(\mathcal{R})
\end{aligned} \tag{3.0.71}$$

Corollary E.41. Let $(V,\mathfrak{q})$ be a finite dimensional quadratic vector space over $\mathbb{R}$. Then the twisted conjugation induces a well-defined and surjective group homomorphism onto the group of radical preserving orthogonal automorphisms of $(V,\mathfrak{q})$:

$$\rho : \mathrm{Pin}(V,\mathfrak{q}) \to \mathrm{O}_{\mathcal{R}}(V,\mathfrak{q}), \tag{3.0.72}$$

with kernel:

$$\ker\left(\rho : \underset{\sim\,\text{in}}{\mathrm{Pin}}(V,\mathfrak{q}) \to \mathrm{O}_{\mathcal{R}}(V,\mathfrak{q})\right) = \{\pm 1\}. \tag{3.0.73}$$

Correspondingly, for the $\mathrm{Spin}(V,\mathfrak{q})$ group. In short, we have short exact sequences:

$$\begin{aligned}
1 \longrightarrow \{\pm 1\} &\xrightarrow{\text{incl}} \mathrm{Pin}(V,\mathfrak{q}) \xrightarrow{\rho} \mathrm{O}_{\mathcal{R}}(V,\mathfrak{q}) \longrightarrow 1, \\
1 \longrightarrow \{\pm 1\} &\xrightarrow{\text{incl}} \mathrm{Spin}(V,\mathfrak{q}) \xrightarrow{\rho} \mathrm{SO}_{\mathcal{R}}(V,\mathfrak{q}) \longrightarrow 1.
\end{aligned} \tag{3.0.74}$$

Definition 3.0.75 (Spin group [Gallier(2014)])

Every Clifford algebra $\mathrm{Cl}(\Phi)$ possesses a canonical anti-automorphism $t : \mathrm{Cl}(\Phi) \to \mathrm{Cl}(\Phi)$ satisfying the properties

$$t(xy) = t(y)t(x), \quad t \circ t = \mathrm{id}, \quad \text{and} \quad t(i(v)) = i(v), \tag{3.0.76}$$

for all $x, y \in \mathrm{Cl}(\Phi)$ and all $v \in V$. Furthermore, such an anti-automorphism is unique.

Every Clifford algebra $\mathrm{Cl}(\Phi)$ has a unique canonical automorphism $\alpha : \mathrm{Cl}(\Phi) \to \mathrm{Cl}(\Phi)$ satisfying the properties

$$\alpha \circ \alpha = \mathrm{id}, \quad \text{and} \quad \alpha(i(v)) = -i(v), \tag{3.0.77}$$

for all $v \in V$.

First, we define conjugation on a Clifford algebra $\mathrm{Cl}(\Phi)$ as the map

$$x \mapsto \bar{x} = t(\alpha(x)) \text{ for all } x \in \mathrm{Cl}(\Phi). \tag{3.0.78}$$

Given a finite dimensional vector space V and a quadratic form Φ on V, the Clifford group of Φ is the group

$$\Gamma(\Phi) = \left\{ x \in \mathrm{Cl}(\Phi)^* \mid \alpha(x) v x^{-1} \in V \quad \text{for all } v \in V \right\}. \tag{3.0.79}$$

The map $N : \mathrm{Cl}(Q) \to \mathrm{Cl}(Q)$ given by

$$N(x) = x\bar{x} \tag{3.0.80}$$

is called the norm of $\mathrm{Cl}(\Phi)$.

We also define the group $\Gamma^+(\Phi)$, called the special Clifford group, by

$$\Gamma^+(\Phi) = \Gamma(\Phi) \cap \mathrm{Cl}^0(\Phi). \tag{3.0.81}$$

We define the pinor group $\mathrm{Pin}(p, q)$ as the group

$$\mathrm{Pin}(p, q) = \left\{ x \in \Gamma_{p,q} \mid N(x) = \pm 1 \right\}, \tag{3.0.82}$$

and the spinor group $\mathrm{Spin}(p, q)$ as $\mathrm{Pin}(p, q) \cap \Gamma^+_{p,q}$.

The restriction of $\rho : \Gamma_{p,q} \to \mathbf{GL}(n)$ to the pinor group $\mathrm{Pin}(p, q)$ is a surjective homomorphism $\rho : \mathbf{Pin}(p, q) \to \mathbf{O}(p, q)$ whose kernel is $\{-1, 1\}$, and the restriction of ρ to the spinor group $\mathbf{Spin}(p, q)$ is a surjective homomorphism $\rho : \mathbf{Spin}(p, q) \to \mathbf{SO}(p, q)$ whose kernel is $\{-1, 1\}$.

Remark: According to Atiyah, Bott and Shapiro, the use of the name $\mathrm{Pin}(k)$ is a joke due to Jean-Pierre Serre (Atiyah, Bott and Shapiro [Atiyah et al.(1964)], page 1).

Definition 3.0.83 (Spin group [Fulton and Harris(2013)])

Instead of defining the spin group as the set of products of certain elements of V, it will be convenient to start with a more abstract definition. Set

$$\mathrm{Spin}(Q) = \{x \in C(Q)^{\text{even}} : x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\} . \tag{3.0.84}$$

We see from this definition that $\mathrm{Spin}(Q)$ forms a closed subgroup of the group of units in the (even) Clifford algebra. Any x in $\mathrm{Spin}(Q)$ determines an endomorphism $\rho(x)$ of V by

$$\rho(x)(v) = x \cdot v \cdot x^*, \quad v \in V. \tag{3.0.85}$$

Define a larger subgroup, this time of the multiplicative group of $C(Q)$, by

$$\mathrm{Pin}(Q) = \{x \in C(Q) : x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\} , \tag{3.0.86}$$

and define a homomorphism

$$\rho : \mathrm{Pin}(Q) \to \mathrm{O}(Q), \quad \rho(x)(v) = \alpha(x) \cdot v \cdot x^*, \tag{3.0.87}$$

where $\alpha : C(O) \to C(O)$ is the main involution.

Definition 3.0.88 (Spin group [Wikipedia(2023)])

The pin group $\mathrm{Pin}(V)$ is a subgroup of $\mathrm{Cl}(V)$'s Clifford group of all elements of the form

$$v_1 v_2 \cdots v_k \tag{3.0.89}$$

where each $v_i \in V$ is of unit length: $q(v_i) = 1$.

The spin group is then defined as

$$\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \mathrm{Cl}^{\text{even}} , \tag{3.0.90}$$

where $\mathrm{Cl}^{\text{even}} = \mathrm{Cl}^0 \oplus \mathrm{Cl}^2 \oplus \mathrm{Cl}^4 \oplus \cdots$ is the subspace generated by elements that are the product of an even number of vectors. That is, $\mathrm{Spin}(V)$ consists of all elements of $\mathrm{Pin}(V)$, given above, with the restriction to k being an even number. The restriction to the even subspace is key to the formation of two-component (Weyl) spinors, constructed below.

Definition 3.0.91 (Spin group [nLab(2023)])

The Pin group $\mathrm{Pin}(V; q)$ of a quadratic vector space, is the subgroup of the group of units in the Clifford algebra $\mathrm{Cl}(V, q)$

$$\mathrm{Pin}(V, q) \hookrightarrow \mathrm{GL}_1(\mathrm{Cl}(V, q)) \tag{3.0.92}$$

on those elements which are multiples $v_1 \cdots v_n$ of elements $v_i \in V$ with $q(v_i) = 1$.

The Spin group $\mathrm{Spin}(V, q)$ is the further subgroup of $\mathrm{Pin}(V; q)$ on those elements which are even number multiples $v_1 \cdots v_{2k}$ of elements $v_i \in V$ with $q(v_i) = 1$.

Definition 3.0.93 (Spin group [Dereli et al.(2010)])
The group $\mathrm{Spin}(p,q)$ is defined by

$$\mathrm{Spin}(p,q) = \left\{ v_1 \dots v_m \in C\ell_{p,q} \mid m \in 2\mathbb{Z}^+, v_i = \sum_{j=1}^{p+q} a_{ij} e_j, \langle v_i, v_i \rangle = \mp 1, 1 \leq i \leq m \right\} \tag{3.0.94}$$

Definition 3.0.95 (Degenerate Spin group [Dereli et al.(2010)])
The subset of $C\ell_{p,q,r}$ defined by

$$S_{p,q,r} = \left\{ s\gamma_1 \dots \gamma_{p+q}(1+G) \mid s \in \mathrm{Spin}(p,q), \gamma_i = 1 + e_i \sum_{l=1}^{r} c_{il} f_l, G \in \Lambda(f) \right\} \tag{3.0.96}$$

is a group under the Clifford multiplication where $\langle \cdot, \cdot \rangle$ is a symmetric bilinear form on $\mathbb{R}^{p+q+r}$, $\{e_1, \dots, e_p, e_{p+1}, \dots, e_{p+q}, f_1, \dots, f_r\}$ is the algebra basis for the degenerate Clifford algebra $C\ell_{p,q,r} = C\ell\left(\mathbb{R}^n, \langle\rangle\right)$, $1 \leq i \leq p+q$, $c_{il} \in \mathbb{R}$, and $\Lambda(f)$ is defined by

$$\Lambda(f) = \mathrm{Span}\left\{ f_{k_1} \dots f_{k_j} \mid 1 \leq k_1 < k_2 < \dots < k_j \leq r \right\}. \tag{3.0.97}$$

References

[Atiyah et al.(1964)] Michael F Atiyah, Raoul Bott, and Arnold Shapiro. 1964. Clifford modules. *Topology* 3 (1964), 3–38. (Cited in section 3.0.75)

[Bär(2018)] Christian Bär. 2018. Lecture notes in Spin Geometry. https://www.math.uni-potsdam.de/en/professuren/geometry/teaching/lecture-notes (Cited in section 3.0.33)

[Dereli et al.(2010)] Tekin Dereli, Şahin Koçak, and Murat Limoncu. 2010. Degenerate spin groups as semi-direct products. *Advances in applied Clifford algebras* 20, 3 (2010), 565–573. (Cited in sections 3.0.93 and 3.0.95)

[Dutailly(2018)] Jean Claude Dutailly. 2018. CLIFFORD ALGEBRAS-NEW RESULTS. (2018). (Cited in section 3.0.48)

[Figueroa-O'Farrill(2010)] José Figueroa-O'Farrill. 2010. Spin geometry. *Lecture Notes. University of Edinburgh* (2010). http://mat.uab.cat/~rubio/csa2017/SpinNotes.pdf (Cited in section 3.0.40)

[Fulton and Harris(2013)] William Fulton and Joe Harris. 2013. *Representation theory: a first course*. Vol. 129. Springer Science & Business Media. (Cited in section 3.0.83)

[Gallier(2014)] Jean Gallier. 2014. Clifford algebras, clifford groups, and a generalization of the quaternions. *arXiv preprint arXiv:0805.0311* (2014). (Cited in section 3.0.75)

[Garling(2011)] David JH Garling. 2011. *Clifford algebras: an introduction*. Vol. 78. Cambridge University Press. (Cited in section 3.0.18)

[Hahn(2004)] Alexander J Hahn. 2004. The Clifford Algebra in the Theory of Algebras, Quadratic Forms, and Classical Groups. *Clifford Algebras: Applications to Mathematics, Physics, and Engineering* (2004), 305–322. (Cited in section 3.0.56)

[Hitzer(2012)] Eckhard Hitzer. 2012. Introduction to Clifford's geometric algebra. *Journal of the Society of Instrument and Control Engineers* 51, 4 (2012), 338–350. (Cited in section 3.0.49)

[Jadczyk(2019)] Arkadiusz Jadczyk. 2019. Notes on Clifford Algebras. (2019). (Cited in section 3.0.15)

[Lawson and Michelsohn(2016)] H. Blaine Lawson and Marie-Louise Michelsohn. 2016. *Spin Geometry (Pms-38)*. Vol. 38. Princeton university press. (Cited in section 3.0.1)

[Li(2008)] Hongbo Li. 2008. *Invariant algebras and geometric reasoning*. World Scientific. (Cited in section 3.0.7)

[Liu(2016)] Henry Liu. 2016. Notes for Lie Groups and Representations. https://member.ipmu.jp/henry.liu/notes/f16-lie-groups.pdf (Cited in section 3.0.39)

[Lundholm and Svensson(2009)] Douglas Lundholm and Lars Svensson. 2009. Clifford algebra, geometric algebra, and applications. *arXiv preprint arXiv:0907.5356* (2009). (Cited in section 3.0.42)

[Meinrenken(2009)] Eckhard Meinrenken. 2009. Lecture notes in MAT 1120HF Lie groups and Clifford algebras. https://www.math.utoronto.ca/mein/teaching/LieClifford/clifford.html (Cited in section 3.0.25)

[Miao et al.(2024)] Jiale Miao, Utensil Song, and Eric Wiesser. 2024. mathlib4 PR 9111. https://github.com/leanprover-community/mathlib4/pull/9111 (Cited in sections 2.0.1, 2.0.4, 2.0.8, and 2.0.13)

[nLab(2023)] nLab. 2023. Spin group - nLab. https://ncatlab.org/nlab/show/spin+group (Cited in section 3.0.91)

[Perwass et al.(2009)] Christian Perwass, Herbert Edelsbrunner, Leif Kobbelt, and Konrad Polthier. 2009. *Geometric algebra with applications in engineering*. Vol. 4. Springer. (Cited in section 3.0.14)

[Porteous(1995)] Ian R Porteous. 1995. *Clifford algebras and the classical groups*. Number 50. Cambridge University Press. (Cited in section 3.0.61)

[Renaud(2020)] Pierre Renaud. 2020. Clifford algebras lecture notes on applications in physics. (Cited in section 3.0.44)

[Rosén(2019)] Andreas Rosén. 2019. *Geometric multivector analysis*. Springer. (Cited in section 3.0.65)

[Ruhe et al.(2024)] David Ruhe, Johannes Brandstetter, and Patrick Forré. 2024. Clifford group equivariant neural networks. *Advances in Neural Information Processing Systems* 36 (2024). (Cited in section 3.0.68)

[Sommer(2013)] Gerald Sommer. 2013. *Geometric computing with Clifford algebras: theoretical foundations and applications in computer vision and robotics*. Springer Science & Business Media. (Cited in section 3.0.8)

[Weber(2013)] Brian Weber. 2013. Lecture notes in MATH 651: Lie Algebras. https://www2.math.upenn.edu/~brweber/Courses/2013/Math651/Math651.html (Cited in section 3.0.31)

[Wikipedia(2023)] Wikipedia. 2023. Spin group - Wikipedia. https://en.wikipedia.org/wiki/Spin_group (Cited in section 3.0.88)

[Wikipedia(2024)] Wikipedia. 2024. Clifford algebra - Wikipedia. https://en.wikipedia.org/wiki/Clifford_algebra (Cited in section 3.0.6)

[Woit(2012)] Peter Woit. 2012. Lecture notes in Lie Groups and Representations: Mathematics G4344. https://www.math.columbia.edu/~woit/LieGroups-2012/ (Cited in section 3.0.36)