The human-curated Group-Algebra route

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A finite group becomes linear once its elements are used as a basis. This route starts with that construction, passes through modules, semisimple blocks, and characters, and then follows normal subgroups, computation, and twisted multiplication. The explicit quaternion calculations earlier in these notes are test cases for this general route, not substitutes for it.

Definition 1 (the Group Algebra and its universal extension `MonoidAlgebra.lift` `MonoidAlgebra.lift_single`)
Let k be a commutative ring and G a group. The **Group Algebra** $k[G]$ is the free k-module with basis $[g]$ indexed by $g \in G$, equipped with

$$[g][h] = [gh], \qquad 1 = [1_G], \qquad (\sum_g a_g[g])(\sum_h b_h[h]) = \sum_{g,h} a_g b_h[gh]. \tag{1}$$

The scalar embedding sends a to $a[1_G]$. Thus each group element is a unit of $k[G]$, with inverse $[g^{-1}]$.

If A is a k-algebra and $u : G \to A^\times$ is a group homomorphism, there is a unique k-algebra homomorphism

$$\widetilde{u} : k[G] \longrightarrow A, \qquad \widetilde{u}\left(\sum_g a_g[g]\right) = \sum_g a_g u(g). \tag{2}$$

This universal property extends representations and concrete group maps from basis elements to the whole Group Algebra. See [Sengupta(2010), secs. 3.1–3.2, pp. 39–41].

Theorem 2 (representations are Group-Algebra modules `Representation.asModule` `Representation.ofModule` `Rep.equivalenceModuleMonoidAlgebra`)

Let k be a field, G a group, and V a k-vector space. Giving a linear representation $\rho : G \to \mathrm{GL}_k(V)$ is equivalent to giving a unital left $k[G]$-module structure on V, up to the identity type synonyms used to keep the 2 scalar actions distinct. The action associated to ρ is

$$\Big(\sum_g a_g[g]\Big) \cdot v = \sum_g a_g \rho(g) v. \tag{3}$$

Conversely, restriction of a unital $k[G]$-action to the units $[g]$ recovers ρ. Invariant subspaces are submodules and intertwining linear maps are $k[G]$-linear maps.

Proof. The extension in the Group Algebra and its universal extension supplies the algebra homomorphism $k[G] \to \mathrm{End}_k(V)$. Associativity and the unit law give a module. In the other direction, multiplication by each $[g]$ is invertible, and the module law gives the group law. These operations are inverse. See [Sengupta(2010), sec. 3.2, pp. 40–41]. □

Example 3 (the regular representation reads the basis)

Let G act on $k[G]$ by left multiplication. In the basis $\{[h] : h \in G\}$, an element g sends $[h]$ to $[gh]$; hence its matrix is the permutation matrix of left translation. This is the **left regular representation**. It is faithful because the image of $[1_G]$ records g.

The same basis underlies the coefficient calculations in the real Group Algebra of the quaternion group: there, the quaternion coordinate recovers antisymmetric coefficient pairs and the character coordinates recover symmetric pairs. The regular module is the linear carrier for both the abstract Group Algebra and the explicit 8-coordinate computation. Compare [James and Liebeck(2001), ch. 6, pp. 53–58].

Theorem 4 (Maschke averaging `MonoidAlgebra.Submodule.exists_isCompl`)
Let G be finite and k a field in which $|G|$ is invertible. Every subrepresentation $W \subseteq V$ has a G-stable complement. Consequently every finite-dimensional $k[G]$-module, and $k[G]$ itself, is semisimple.

Proof. Choose a k-linear projection $P : V \longrightarrow W$ and average its conjugates:

$$P_G = \frac{1}{|G|} \sum_{g \in G} \rho(g) P \rho(g)^{-1}. \tag{4}$$

Then P_G remains the identity on W and commutes with G. Its kernel is a G-stable complement. Division by $|G|$ requires the stated invertibility hypothesis. This is Maschke's theorem as proved in [Sengupta(2010), thm. 3.5.1, pp. 44–46]. □

Theorem 5 (Wedderburn blocks over an algebraically closed field `exists_algEquiv_pi_matrix_conjClasses`)

Let G be finite and k an algebraically closed field whose characteristic does not divide $|G|$. There are positive integers d_C, indexed by the conjugacy classes C of G, such that

$$k[G] \cong_{k\text{-alg}} \prod_{C \in \mathrm{Conj}(G)} M_{d_C}(k), \qquad \sum_C d_C^2 = |G|. \tag{5}$$

The class labels are an indexing choice: the statement does not canonically pair a particular conjugacy class with a particular irreducible module.

Maschke semisimplicity followed by Artin–Wedderburn gives the stated product. Lux and Pahlings state the results in [Lux and Pahlings(2010), thms. 1.5.5–1.5.6, pp. 56–57].

The dimension identity can also be read from the regular representation, where an irreducible module occurs with multiplicity equal to its dimension; compare [James and Liebeck(2001), thms. 11.9 and 11.12, pp. 100–101].

Definition 6 (characters, class functions, and the character table `basisOfIrreducibleCharacters` `characterTable` `card_inv_mul_sum_characterTable_mul_characterTable_inv` `card_conjClass_mul_sum_characterTable_mul_characterTable_inv`)

For a finite-dimensional representation ρ over a characteristic-zero field, its character is $\chi_\rho(g) = \mathrm{tr}(\rho(g))$. Trace is unchanged by conjugation, so χ_ρ is a class function. A **character table** places the irreducible characters in rows and the conjugacy classes in columns. Tau Ceti identifies class functions with functions on conjugacy classes in `TauCeti.ClassFunction.equivConjClasses`.

Over $\mathbb{C}$, the irreducible characters form an orthonormal basis of the class functions. Thus the table is square: the number of irreducible characters equals the number of conjugacy classes. This also counts the Wedderburn blocks and gives the dimension of the center. See [James and Liebeck(2001), thm. 16.4, pp. 159–166].

Theorem 7 (primitive central idempotents as block projectors `primitiveCentralIdempotent` `asAlgebraHom_primitiveCentralIdempotent` `primitiveCentralIdempotent_mul_primitiveCentralIdempotent`)

Over a splitting field of characteristic not dividing $|G|$, an irreducible character χ defines the central element

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})[g]. \tag{6}$$

It is a nonzero primitive central idempotent. On a simple module with character ψ, it acts as the identity if $\psi = \chi$ and as zero otherwise. Distinct e_χ are therefore orthogonal block projectors. See [Lux and Pahlings(2010), thm. 2.1.6 and cor. 2.1.7, pp. 88–90]. A constructed family of such projectors does not yield a complete block decomposition until a sum-to-1 statement is also known.

Theorem 8 (the Frobenius–Schur trichotomy `frobeniusSchurIndicator_eq_one_or_eq_zero_or_eq_neg_one` `frobeniusSchurIndicator_eq_one_iff` `frobeniusSchurIndicator_eq_neg_one_iff` `frobeniusSchurIndicator_eq_zero_iff`)

Let ρ be an irreducible finite-dimensional representation of a finite group over an algebraically closed field of characteristic 0. Its Frobenius–Schur indicator

$$\nu_2(\rho) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g^2) \tag{7}$$

takes exactly 1 of the values 1, 0, and −1. The value 1 is equivalent to the existence of a nondegenerate invariant symmetric bilinear form; −1 is equivalent to a nondegenerate invariant alternating form; and 0 is equivalent to the absence of a nonzero invariant bilinear form. See [James and Liebeck(2001), def. 23.13, the proof of thm. 23.14, and thm. 23.16, pp. 273–277]. James and Liebeck state the alternatives using nonzero invariant forms; for an irreducible representation, a nonzero invariant symmetric or alternating form has zero radical and is therefore nondegenerate.

This trichotomy classifies the invariant-form behavior of the complex irreducible representation. It does not by itself compute a Schur index or a complete decomposition after descent to $\mathbb{R}$.

Remark 9 (real types are not Schur indices)
Over $\mathbb{R}$, a simple finite-dimensional block may instead have division algebra $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The Frobenius–Schur trichotomy in the Frobenius–Schur trichotomy detects invariant-form type, but it is not a complete theorem about a character's field of values or Schur index. The explicit decomposition

$$\mathbb{R}[Q_8] \cong \mathbb{R}^4 \times \mathbb{H} \tag{8}$$

in the real Group Algebra of the quaternion group is a complete real example, not an application of the algebraically closed formula. For Schur indices and invariant forms, see [Lux and Pahlings(2010), sec. 2.9, pp. 164–172].

Definition 10 (conjugate constituents and the inertia subgroup)
Let $N \trianglelefteq G$, let V be a G-representation, and let $W \subseteq V|_N$ be an irreducible N-subrepresentation. For $g \in G$, the **conjugate representation** , ${}^gW$ has the same vector space and action

$$n \cdot_g w = (g^{-1}ng) \cdot w. \tag{9}$$

The **inertia subgroup** of W is

$$I_G(W) = \{g \in G : {}^gW \cong W\}. \tag{10}$$

The isomorphism classes of conjugates form a G-orbit, with stabilizer $I_G(W)$. See [Lux and Pahlings(2010), sec. 3.6, pp. 222–225].

Theorem 11 (the single orbit in Clifford restriction `isSemisimpleRepresentation_comp_subtype` `exists_isAtom_forall_nonempty_linearEquiv_conjSubrep`)
Let $N \trianglelefteq G$ be finite, and work over a field for which the relevant finite-group modules are semisimple. If V is an irreducible G-representation, then the irreducible constituents of $V|_N$ lie in 1 G-orbit under conjugation. Equivalently, after choosing 1 constituent W, every other constituent is isomorphic to some , ${}^gW$.

This is the orbit assertion in Clifford theory. It does not by itself say that all constituents occur with the same multiplicity, nor does it construct extensions across the inertia subgroup. Compare [Lux and Pahlings(2010), thm. 3.6.2(a), pp. 223–225].

Theorem 12 (Clifford homogeneity and the semidirect route)
Let $N \trianglelefteq G$ and let V be an irreducible complex representation of G. If $W_1, \ldots, W_r$ represent the distinct conjugates of an irreducible constituent of $V|_N$, then for 1 positive integer e,

$$V|_N \cong e(W_1 \oplus \cdots \oplus W_r). \tag{11}$$

Thus restriction is homogeneous along a single orbit. The common multiplicity is an additional assertion beyond the single orbit in Clifford restriction. This is Clifford's theorem, [James and Liebeck(2001), thm. 20.8, pp. 216–217].

The inertia subgroup is the intermediate group where one studies whether W extends, possibly projectively, before inducing to G. For semidirect products, this reduces the problem to the action of the complement. For $Q_8 \rtimes C_3 \cong 2T$, it connects the normal quaternion subgroup to the skew-Group-Algebra model in two routes from binary tetrahedral symmetry. This theorem does not supply the full character table of $2T$.

Example 13 (semidirect products make Clifford theory concrete)
Suppose $G = N \rtimes H$. Conjugation by H permutes the irreducible constituents of $V|_N$. Their stabilizers are the inertia subgroups, so the orbit and homogeneity statements in the single orbit in Clifford restriction and Clifford homogeneity and the semidirect route reduce the search for irreducible G-representations to stabilizer data, projective extension, and induction. Clifford theory thereby gives a character-theoretic route for semidirect products.

For $Q_8 \rtimes C_3 \cong 2T$, the action and ordinary skew-Group-Algebra realization are recorded in the central split of the real Group Algebra of 2T and two routes from binary tetrahedral symmetry. They supply the normal-subgroup and algebraic input to this route, but do not give the full character table of $2T$.

Remark 14 (from a presentation to checked representation data)
A finite presentation $\langle x_1, \ldots, x_m \mid r_1, \ldots, r_s \rangle$ gives a compact input for algorithms that enumerate cosets, compute conjugacy classes, and build character data. Sims develops the algorithms and the conditions under which such computations terminate; Lux and Pahlings place them inside computational representation theory.

The output has 3 possible evidential strengths. A transcript with versioned inputs is reproducible. A compact certificate, such as matrices satisfying the relations together with independently checked completeness identities, can be verified without trusting the search. An unrecorded software answer is neither. GAP is therefore a discovery and calculation tool; its output becomes mathematics here only when the decisive relations and completeness checks are visible. See [Sims(1994), ch. 1] and [Lux and Pahlings(2010), secs. 1.1 and 4.2].

The route is

$$\begin{aligned} \text{finite presentation} &\longrightarrow \text{computed candidates} \\ &\longrightarrow \text{checked relations and completeness.} \end{aligned} \tag{12}$$

The arrows distinguish generation from certification rather than describing a project workflow.

Definition 15 (factor sets and twisted Group Algebras `of_mul_of` `lift` `isProjectiveRepEquivAlgHom`)
Let k be a field, G a group, and $\alpha : G \times G \to k^\times$ a normalized 2-cocycle:

$$\alpha(1, g) = \alpha(g, 1) = 1, \qquad \alpha(g, h)\alpha(gh, \ell) = \alpha(h, \ell)\alpha(g, h\ell). \tag{13}$$

The **twisted Group Algebra** $k_\alpha[G]$ has basis u_g and multiplication

$$u_g u_h = \alpha(g, h) u_{gh}. \tag{14}$$

The cocycle equation is exactly the associativity condition.

A projective representation with factor set α, namely operators T_g satisfying $T_g T_h = \alpha(g, h) T_{gh}$, extends uniquely to an algebra map $k_\alpha[G] \to \mathrm{End}_k(V)$; conversely an algebra map gives such a projective representation on its basis operators. Cohomologous cocycles rescale the basis and give isomorphic twisted algebras. See [Conlon(1964), the introduction and sec. 1, pp. 152–155].

Remark 16 (the qualified Clifford bridge)
A generalized Clifford algebra becomes a twisted group algebra only after its parameters are fixed. Let k contain a primitive n-th root of unity ω, assume $\operatorname{char} k \nmid n$, and choose $q_1, \ldots, q_m \in k^\times$. The algebra generated by $e_1, \ldots, e_m$ with

$$e_i^n = q_i, \qquad e_i e_j = \omega e_j e_i \quad (j < i) \tag{15}$$

is isomorphic to a twisted group algebra of $(\mathbb{Z}/n\mathbb{Z})^m$ for an explicit 2-cocycle. The monomials in the e_i correspond to its twisted basis. See [Cheng et al.(2019), sec. 2.3 and prop. 2.1, pp. 3–4].

This is a qualified bridge, not an identification of every Clifford algebra with an ordinary Group Algebra. The field, root of unity, grading, parameters, and twisting are part of the theorem. It is also different from Clifford theory in the single orbit in Clifford restriction and Clifford homogeneity and the semidirect route, which concerns restriction of representations to normal subgroups.

Remark 17 (the inertia obstruction is projective)
For a constituent W with inertia subgroup $I = I_G(W)$, choose intertwiners $T_i : W \to {}^iW$. Their composites need not satisfy the group law strictly. Instead one obtains scalars

$$T_i T_j = \alpha(i, j) T_{ij}. \tag{16}$$

Associativity makes α a factor set. Changing the intertwiners changes α by a coboundary. The resulting cohomology class measures the obstruction to replacing the projective action by an ordinary action.

Twisted Group Algebras therefore enter the Clifford-theory route. The ordinary skew Group Algebra in two routes from binary tetrahedral symmetry corresponds to trivial twisting; it should not be identified with a general $k_\alpha[G]$. The extension and induction analysis is developed in [Lux and Pahlings(2010), secs. 3.6–3.7, pp. 222–240].

Proposition 18 (the canonical grading of a Group Algebra `AddMonoidAlgebra.grade.isInternal` `AddMonoidAlgebra.grade.gradedAlgebra` `AddMonoidAlgebra.single_mem_grade`)

For each $g \in G$, let $A_g = k[g]$ be the 1-dimensional subspace spanned by the basis element $[g]$. Then

$$k[G] = \bigoplus_{g \in G} A_g, \qquad A_g A_h \subseteq A_{gh}. \tag{17}$$

Thus $k[G]$ is canonically graded by the group G. In Lean's additive graded-algebra interface this statement is expressed after passing from the multiplicative index G to its additive copy; each single basis element lies in its corresponding grade.

The basis and multiplication law in the Group Algebra and its universal extension give the decomposition and product containment directly; compare the construction in [Sengupta(2010), secs. 3.1–3.2, pp. 39–41]. It is distinct from the Hopf structure in the standard Hopf structure on a Group Algebra: the grading records where products land, while the coalgebra maps record how a basis element is copied and inverted.

Proposition 19 (the standard Hopf structure on a Group Algebra `MonoidAlgebra.instHopfAlgebra` `MonoidAlgebra.counit_single` `MonoidAlgebra.comul_single` `MonoidAlgebra.antipode_single`)

Let k be a commutative ring and G a group. The Group Algebra $k[G]$ is a Hopf algebra with structure determined on basis elements by

$$\Delta([g]) = [g] \otimes [g], \qquad \epsilon([g]) = 1, \qquad S([g]) = [g^{-1}]. \tag{18}$$

The comultiplication and counit are algebra homomorphisms; the antipode extends inversion and satisfies the 2 convolution identities.

Proof. Each formula respects multiplication on basis elements. For example, $\Delta([gh]) = [gh] \otimes [gh] = ([g] \otimes [g])([h] \otimes [h])$. The counit calculation is similar. Finally $[g][g^{-1}] = [1] = [g^{-1}][g]$, which gives both antipode identities on the basis and hence by linearity on all of $k[G]$. This group-algebra Hopf structure appears, for example, in [Broué(2024), ex. 18.3.3(1), pp. 500–501]. □

Remark 20 (what the real quaternion example settles)
The general route has several distinct endpoints. Over an algebraically closed field, Wedderburn blocks and the character table organize all simple representations. Over $\mathbb{R}$, division-algebra type and descent data enter. For a normal subgroup, Clifford theory organizes restriction and induction. With a nontrivial factor set, projective representations are modules over a twisted algebra.

The quaternion calculation in the real Group Algebra of the quaternion group closes 1 bounded endpoint: it gives an explicit real-algebra equivalence $\mathbb{R}[Q_8] \cong \mathbb{R}^4 \times \mathbb{H}$, and Real blocks of the quaternion Group Algebra identifies its center. It does not prove a general real Wedderburn theorem, the characters of $2T$, or the Clifford-theory extension step. Those are separate statements and do not follow from the worked example.

Remark 21 (the Hessian-group outlook)
Wilson's proposed route begins with

$$Q_8 \rtimes C_3 \cong 2T \tag{19}$$

and then studies the iterated semidirect product $(G_{27} \rtimes Q_8) \rtimes C_3$, with the indicated actions part of the data. Its central scalar subgroup $C_3 \leq G_{27}$ has quotient the Hessian group of order 216, so the larger group is a triple cover of that quotient. The name alone would not choose either semidirect action. See [Wilson(2024), sec. 2.2, pp. 4–5].

The comparison with $U(1) \times SU(2) \times SU(3)$ is an attributed finite-model proposal, not an isomorphism or a canonical physical interpretation. To become a mathematical correspondence it would need typed representations and homomorphisms, the preserved forms or observables, and a rule connecting them to physical data. Hamilton's geometric-algebra models in [Hamilton(2023)] and [Hamilton and McMaken(2023)] provide a comparison motivation, not independent confirmation of this finite-group construction.

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