Models and routes from binary tetrahedral structure

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The concrete, internal, external, and abstract models of $2T$ retain different data. Their comparison opens a Group-Algebra route through the semidirect factors and a representation-theoretic route through the McKay correspondence.

Remark 1 (what the models of 2T retain)
The same binary tetrahedral symmetry now has several models. They answer different questions.

- The **concrete model** $T \le \mathbb{H}^\times$ retains quaternion coordinates, multiplication, conjugation, and norm. It is the model for explicit calculations.
- The **internal semidirect-product structure** retains the same group T, but also marks the subgroups $Q, C \le T$. The facts

$$T = QC, \qquad Q \trianglelefteq T, \qquad Q \cap C = \{1\} \tag{1}$$

  say that every element of T has a unique factorization qc. They expose how the two parts fit inside the concrete group.
- The **external model** $Q \rtimes_\alpha C$ replaces a quaternion by a pair (q, c). It retains the factors and records their interaction in the conjugation action $\alpha(c)(q) = cqc^{-1}$. Its multiplication is

$$(q, c)(q', c') = \big(q\alpha(c)(q'), cc'\big). \tag{2}$$

- An **abstract model** can specify the binary tetrahedral group without quaternion coordinates. This is convenient when only its group structure is needed, but coordinates and named factors must be restored by additional maps.

The multiplication map

$$\mu : Q \rtimes_\alpha C \longrightarrow T, \qquad (q, c) \longmapsto qc \tag{3}$$

is an isomorphism by the internal semidirect-product theorem. Indeed,

$$\mu\big((q, c)(q', c')\big) = q(cq'c^{-1})cc' = qcq'c' = \mu(q, c)\mu(q', c'). \tag{4}$$

Thus the external multiplication is not an analogy: it is exactly the concrete multiplication written in factor coordinates. The two carriers are still different, so a theorem or construction passes between them only through an explicit isomorphism. If it refers to the named factors or their action, the transport must also record the corresponding compatibility.

The internal and external constructions are developed in [Fré(2023), sec. 4.2.13, pp. 62–63] and [Isaev and Rubakov(2018), sec. 1.4.2, pp. 58–61]. The concrete identification of $2T$ with the Hurwitz unit group and $Q_8 \rtimes \mathbb{Z}/3\mathbb{Z}$ is given in [Voight(2021), sec. 11.2.4, p. 168].

Remark 2 (two routes from binary tetrahedral symmetry)
The semidirect-product model and the quaternion model open two different routes from the same group. One rewrites the Group Algebra in interacting factor data. The other places the group inside $\mathrm{SU}(2)$ and leads to the McKay graph.

2.1 The Group-Algebra route

Extend the conjugation action $\alpha : C \to \mathrm{Aut}(Q)$ linearly to real-algebra automorphisms

$$\overline{\alpha}_c : \mathbb{R}[Q] \longrightarrow \mathbb{R}[Q]. \tag{2.1.1}$$

On the vector space $\mathbb{R}[Q] \otimes_{\mathbb{R}} \mathbb{R}[C]$, write a pure tensor as $a \# c$ and define

$$(a \# c)(b \# d) = a\,\overline{\alpha}_c(b) \# cd, \tag{2.1.2}$$

extending bilinearly. This convention defines the **skew Group Algebra** $\mathbb{R}[Q] \rtimes_{\overline{\alpha}} C$; it is also a crossed product with trivial twisting cocycle.

The basis map

$$[q] \# c \longmapsto [qc] \tag{2.1.3}$$

respects multiplication by the calculation in what the models of 2T retain, and unique factorization makes it a bijection. Hence

$$\mathbb{R}[Q] \rtimes_{\overline{\alpha}} C \cong_{\mathbb{R}\text{-alg}} \mathbb{R}[T]. \tag{2.1.4}$$

This is the Group-Algebra form of $T \cong Q \rtimes_{\alpha} C$. The extension of group maps to Group-Algebra maps follows the convention in [Sengupta(2010), secs. 3.1–3.2, pp. 39–41].

A module over this skew Group Algebra can be read as an $\mathbb{R}[Q]$-module M together with a linear action of C satisfying

$$c \cdot (a \cdot m) = \overline{\alpha}_c(a) \cdot (c \cdot m). \tag{2.1.5}$$

The external model therefore exposes the compatibility needed to assemble representations from the two factors. It does not by itself decompose $\mathbb{R}[T]$ into simple blocks. That requires further representation-theoretic input.

2.2 The McKay route

Under the standard matrix realization of the unit quaternions as $\mathrm{SU}(2)$, the concrete group T becomes a finite subgroup of $\mathrm{SU}(2)$. Let τ_3 be its faithful 2-dimensional complex representation. The McKay graph has one vertex for each irreducible complex representation τ_i; the number of edges from τ_i to τ_j is the multiplicity of τ_j in

$$\tau_3 \otimes \tau_i. \tag{2.2.1}$$

For the binary tetrahedral group this graph is the affine Dynkin diagram $\widetilde{E}_6$. The character table, the seven irreducible representations, and the tensor-product calculation are given in [Stekolshchik(2008), table A.12, prop. A.10, and ex. A.11, pp. 172–174].

This graph connects the representation theory of $2T$ with quivers and the ADE classification. The same source relates a binary polyhedral subgroup $G \leq \mathrm{SU}(2)$ to the quotient $\mathbb{C}^2/G$, its invariant algebra, and its Kleinian singularity in [Stekolshchik(2008), secs. A.3–A.4, pp. 158–161]. These are routes toward quotient geometry and related orbifold constructions. They are not consequences of the semidirect-product theorem alone.

Neither route assigns physical meaning to Q, C, an algebra block, or a vertex of $\widetilde{E}_6$. They supply mathematical structures on which a physical correspondence could be stated and tested. A proposed correspondence must still identify the action, representation, preserved structure, and physical interpretation.

References

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