Foundational Group Algebras (for) Physics

Utensil Song

July 28, 2026

These are the accompanying mathematical notes for the experimental research project FGAP. FGAP is read "F-gap," for the foundational gap between established mathematical tools and open questions about physical structures, especially beyond the Standard Model. It is also a coined acronym for "Foundational Group Algebras (for) Physics."

The project studies mathematical structures that may help us understand physical structures more deeply. It does not construct or claim a physical theory. The current storyline has been strongly shaped by agent exploration of one spike-test slice of that research domain. It follows the project's approach: turn a structural hypothesis into a small, source-grounded vertical slice, then use explicit calculations and formalization to expose its assumptions.

This slice begins with Hamilton's Quaternions and the quaternionic realization of the binary tetrahedral group, because they give a small and explicit route into finite real Group Algebras. For the human-curated storyline of the notes, see [spin-0010].

For the mathematics, we follow the Hurwitz-unit treatment in [Voight(2021), ch. 11]. We keep the mathematics and the physical suggestions separate.

Remark 1 (why begin with 2T?)
The binary tetrahedral group $2T$ is small enough for explicit calculations, yet it connects finite symmetry, representations, real Group Algebras, and the quaternionic structures familiar from spin mathematics. Its concrete realization by Hurwitz units, and their identification with $2T$, are described in [Voight(2021), sec. 11.2, pp. 166–168], while the relation between representations and Group-Algebra modules is developed in [Sengupta(2010), secs. 3.1–3.3, pp. 39–42]. Wilson uses order-3 automorphisms of Q_8 to discuss possible physical interpretations in [Wilson(2021), sec. 4.3, pp.

12–13, v5]. He later adjoins an abstract generator f of order 3 to Q_8 to form $2T$ in [Wilson(2021), sec. 6.1, p. 18, v5].

Any physical identification remains a hypothesis. A proposed correspondence must specify an action and representation, an algebra map, any preserved structure or form, a unit-group choice where applicable, and the intended physical meaning. Resemblance alone is not a model.

2 Quaternion and Hurwitz foundations

We first introduce Hamilton's Quaternions, the Hurwitz order, and a distinguished unit. Its order and conjugation action provide the concrete calculations used by the group constructions in the next two sections.

Definition 2.1 (Hamilton's Quaternions)

The algebra of real **quaternion**s is the 4-dimensional real vector space

$$\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k. \tag{2.1}$$

Thus every quaternion can be written uniquely as

$$q = a + bi + cj + dk, \qquad a, b, c, d \in \mathbb{R}. \tag{2.2}$$

Addition and real scalar multiplication are coordinatewise. Multiplication is extended bilinearly from 1 as the multiplicative identity and the table

$$\begin{gathered} i^2 = j^2 = k^2 = ijk = -1, \\ ij = k, \quad jk = i, \quad ki = j, \\ ji = -k, \quad kj = -i, \quad ik = -j. \end{gathered} \tag{2.3}$$

This makes $\mathbb{H}$ an associative, noncommutative real algebra. In Voight's notation, $\mathbb{H} = (-1, -1 \mid \mathbb{R})$; see [Voight(2021), def. 2.2.1 and ex. 2.2.3].

More generally, when F is a field of characteristic different from 2 and $a, b \in F^\times$, the symbol

$$(a, b \mid F) \tag{2.4}$$

denotes the quaternion algebra over F generated by elements i, j with

$$i^2 = a, \qquad j^2 = b, \qquad ji = -ij. \tag{2.5}$$

The two slots record the scalars assigned to the squares of the two generators. With $k = ij$, the elements $1, i, j, k$ form an F-basis. Thus the two entries $-1, -1$ in $(-1, -1 \mid \mathbb{R})$ say precisely that $i^2 = -1$ and $j^2 = -1$; the anticommutation relation then gives $k^2 = -1$.

For $q = a + bi + cj + dk$, its **quaternion conjugate** and **reduced norm** are

$$\overline{q} = a - bi - cj - dk, \qquad \operatorname{nrd}(q) = q\overline{q} = a^2 + b^2 + c^2 + d^2. \tag{2.6}$$

We also have $\overline{q}q = \operatorname{nrd}(q)$. If $q \neq 0$, then $\operatorname{nrd}(q) > 0$ and

$$q^{-1} = \frac{\overline{q}}{\operatorname{nrd}(q)}. \tag{2.7}$$

So every nonzero quaternion is invertible. This is the standard involution and reduced norm specialized to Hamilton's Quaternions; see [Voight(2021), secs. 3.1–3.3].

The corresponding Mathlib type is `Quaternion`.

Definition 2.2 (the Hurwitz order and its units)
Inside the rational quaternion algebra $B = (-1, -1 \mid \mathbb{Q})$, let $k = ij$ and

$$\omega = \frac{-1 + i + j + k}{2}. \tag{2.8}$$

The **Lipschitz order** and **Hurwitz order** are the $\mathbb{Z}$-lattices

$$\begin{aligned} L &= \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k, \\ \mathcal{O} &= \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}\omega. \end{aligned} \tag{2.9}$$

Both are subrings containing 1. The order $\mathcal{O}$ contains L with index 2; it is the unique order properly containing L, and it is maximal. See [Voight(2021), sec. 11.1, esp. lem. 11.1.2].

A **Hurwitz unit** is a unit of $\mathcal{O}$. Equivalently, it is an element $q \in \mathcal{O}$ whose reduced norm is 1. Voight calculates that $\mathcal{O}^\times$ has 24 elements in [Voight(2021), sec. 11.2, pp. 166–168]. The particular unit ω will give the cyclic factor used below.

Remark 2.3 (why the Hurwitz order is introduced)
Hurwitz developed integral quaternions in 1919. Starting from the Lipschitz order L, the question is not merely which quaternions have integral coordinates, but which order has the better arithmetic structure. The Lipschitz order is not maximal. Voight compares this with enlarging $\mathbb{Z}[\sqrt{-3}]$ to the Eisenstein integers. Since $a = i + j + k$ satisfies $a^2 = -3$, the analogous element

$$\omega = \frac{-1 + a}{2} \tag{2.10}$$

satisfies $\omega^2 + \omega + 1 = 0$. Adjoining it enlarges L to the maximal Hurwitz order $\mathcal{O}$. See [Voight(2021), sec. 11.1, pp. 165–166].

This enlargement supplies more than a convenient lattice. Its 24 units form the binary tetrahedral group, and conjugation by ω cyclically permutes the quaternion units i, j, k. The same order also supports a norm-Euclidean algorithm. These features make the Hurwitz order a small meeting point of quaternion arithmetic, finite-group structure, and explicit calculation; see [Voight(2021), secs. 11.2–11.3, pp. 166–169].

Lemma 2.4 (a Hurwitz unit of order 3)
Let

$$a = i + j + k, \qquad \omega = \frac{-1 + i + j + k}{2} = \frac{-1 + a}{2}. \tag{2.11}$$

Then ω is a Hurwitz unit of order 3. This choice appears in [Voight(2021), p. 165].

The mixed terms in a^2 cancel in pairs:

$$\begin{aligned} a^2 &= i^2 + j^2 + k^2 + (ij + ji) + (jk + kj) + (ki + ik) \\ &= -3. \end{aligned} \tag{2.12}$$

It follows that

$$\begin{aligned} \omega^2 &= \frac{(-1 + a)^2}{4} \\ &= \frac{1 - 2a + a^2}{4} \\ &= \frac{-1 - a}{2}. \end{aligned} \tag{2.13}$$

Hence $\omega^2 + \omega + 1 = 0$, and multiplication by $\omega - 1$ gives

$$\omega^3 = 1. \tag{2.14}$$

Thus ω is invertible, has inverse $\omega^{-1} = \omega^2 = (-1 - i - j - k)/2$, and has order 3 because $\omega \neq 1$.

Convention 2.5 (left conjugation)
For a nonzero quaternion q, we use **left conjugation** to mean

$$q \triangleright x = qxq^{-1}. \tag{2.15}$$

See the **remark on the conjugation calculation** for why the direction is recorded explicitly.

Quaternions can also be obtained from suitable Clifford Algebras; see Notes on Clifford Algebras. Here we use the multiplication in **Hamilton's Quaternions** directly.

Example 2.6 (conjugation by a Hurwitz unit)
Consider left conjugation by the Hurwitz unit $\omega = (-1 + i + j + k)/2$. We have

$$\omega i \omega^{-1} = k, \qquad \omega j \omega^{-1} = i, \qquad \omega k \omega^{-1} = j. \tag{2.16}$$

Thus the cycle is $i \mapsto k \mapsto j \mapsto i$. Voight says that conjugation cyclically rotates these units in [Voight(2021), sec. 11.2.4, p. 168]; the calculation here fixes the direction. Wilson discusses an abstract order-3 automorphism cycling the quaternion generators in [Wilson(2021), sec. 4.3, pp. 12–13, v5]. That discussion does not select the exact quaternion or fix the oriented cycle calculated here.

It is enough to calculate

$$\omega i = k\omega, \qquad \omega j = i\omega, \qquad \omega k = j\omega. \tag{2.17}$$

For example,

$$\omega i = \frac{-1 - i + j - k}{2} = k\omega. \tag{2.18}$$

Right multiplication by ω^{-1} proves the three conjugation equations.

Remark 2.7 (what the conjugation calculation gives)
Saying that conjugation cyclically permutes i, j, k does not by itself tell us which cycle is meant. Left conjugation by ω gives the cycle in the **conjugation example**. Conjugation by ω^{-1} gives the inverse cycle. This is why we record the three equations instead of only saying "cyclically permutes."

The calculation supplies the first concrete group action used for the binary tetrahedral group. The next notes restrict this action to the quaternion subgroup and then use it in the semidirect-product construction. The three equations alone do not identify the complete set of 24 Hurwitz units with that group.

In Lean, the three equations give the computational interface to Mathlib's `Quaternion` type. The abstract action is then recovered from this concrete calculation.

3 Quaternion and order-three subgroups

We now separate two small groups inside the nonzero quaternions. The first is the quaternion group generated by i and j. The second is generated by the Hurwitz unit ω. Conjugation by ω then tells us how the second group acts on the first.

This is the concrete input for a later construction of the binary tetrahedral group. We stop before that construction here: the present notes only define the two factors and calculate their action.

Definition 3.1 (group action)
A **left action** of a group G on a set X is a map

$$G \times X \longrightarrow X, \qquad (g, x) \longmapsto g \cdot x \tag{3.1}$$

such that

$$1 \cdot x = x, \qquad g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x. \tag{3.2}$$

Equivalently, an action is a group homomorphism

$$\alpha : G \longrightarrow \mathrm{Sym}(X), \tag{3.3}$$

where $\mathrm{Sym}(X)$ is the group of permutations of X. We use left actions throughout these notes. See [Woit(2024), sec. 1.3.2, pp. 7–9].

If $X = H$ is a group and each permutation preserves multiplication, the action is a homomorphism

$$\alpha : G \longrightarrow \mathrm{Aut}(H). \tag{3.4}$$

Its **kernel** is $\ker \alpha$. The action is **faithful** when α is injective.

Suppose G acts on both X and Y. A map $f : X \to Y$ is **equivariant** if

$$f(g \cdot x) = g \cdot f(x) \tag{3.5}$$

for every $g \in G$ and $x \in X$. Equivariance is how an action passes between two different models of the same object.

Definition 3.2 (the quaternion group inside Hamilton's Quaternions)
Inside the group $\mathbb{H}^\times$ of nonzero **quaternions**, let

$$Q = \{\pm 1, \pm i, \pm j, \pm k\}. \tag{3.6}$$

The multiplication table for Hamilton's Quaternions shows that this set is closed under multiplication and inverses. It is therefore a subgroup of $\mathbb{H}^\times$.

The elements i and j satisfy

$$i^4 = 1, \qquad i^2 = j^2 = -1, \qquad ji = i^{-1}j. \tag{3.7}$$

These are the usual relations for the quaternion group. Hence Q is a concrete copy of Q_8. Voight identifies the same 8 units in [Voight(2021), sec. 11.2, p. 166].

Here Q always means this particular subgroup of the quaternions. An abstract quaternion group will be introduced only when a distinction between models is useful.

Lemma 3.3 (normal forms in the quaternion subgroup)
Every element of Q has a unique normal form

$$i^a j^b, \qquad 0 \le a < 4, \quad 0 \le b < 2. \tag{3.8}$$

Consequently,

$$Q = \langle i, j \rangle = \{i^a j^b : 0 \le a < 4,\ 0 \le b < 2\} \tag{3.9}$$

and Q has 8 elements.

Proof. The relation $ji = i^{-1}j$ moves each occurrence of j to the right. The relations $i^4 = 1$ and $j^2 = i^2$ then reduce every word in i, j to the stated form.

The 8 resulting quaternions are

$$\begin{matrix} 1 & i & -1 & -i \\ j & k & -j & -k. \end{matrix} \tag{3.10}$$

Their coordinates in $\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ are distinct, so the normal forms are unique. □

The two-row display will also be useful when the concrete subgroup is compared with an abstract presentation of Q_8. See [Voight(2021), sec. 11.2, p. 166].

Lemma 3.4 (the cyclic subgroup generated by the Hurwitz unit)
Let

$$\omega = \frac{-1 + i + j + k}{2}. \tag{3.11}$$

The calculation in a Hurwitz unit of order 3 gives $\omega^3 = 1$ and $\omega \neq 1$. It follows that

$$C = \langle \omega \rangle = \{1, \omega, \omega^2\} \tag{3.12}$$

is a cyclic subgroup of $\mathbb{H}^\times$ of order 3. In particular, $C \cong \mathbb{Z}/3\mathbb{Z}$. This subgroup appears in [Voight(2021), sec. 11.2.4, p. 168].

Proof. Every power of ω reduces modulo 3, so $C \subseteq \{1, \omega, \omega^2\}$. The reverse inclusion is immediate. If $\omega^2 = 1$, multiplying by ω would give $\omega = 1$, contrary to the coordinate formula. Thus the 3 displayed elements are distinct. □

Example 3.5 (the Hurwitz action on the quaternion group)
The subgroup C acts on Q by left conjugation:

$$C \longrightarrow \mathrm{Aut}(Q), \qquad c \longmapsto (q \longmapsto cqc^{-1}). \tag{3.13}$$

Indeed, conjugation by a Hurwitz unit gives

$$\omega i \omega^{-1} = k, \qquad \omega k \omega^{-1} = j, \qquad \omega j \omega^{-1} = i. \tag{3.14}$$

Thus conjugation by the generator preserves all 8 elements of Q. Its conjugation powers preserve Q as well, so every $c \in C$ gives the stated automorphism. The generator's action can be read from the two oriented cycles

$$i \longmapsto k \longmapsto j \longmapsto i, \qquad -i \longmapsto -k \longmapsto -j \longmapsto -i, \tag{3.15}$$

while 1 and -1 are fixed.

Conjugation preserves products:

$$c(q_1 q_2)c^{-1} = (cq_1 c^{-1})(cq_2 c^{-1}). \tag{3.16}$$

It is therefore an automorphism of Q; its inverse is conjugation by c^{-1}. The action law follows from

$$(c_1 c_2)q(c_1 c_2)^{-1} = c_1(c_2 q c_2^{-1})c_1^{-1}. \tag{3.17}$$

This verifies the action directly rather than inferring it from the picture. Voight records the normality of Q and the cyclic rotation in [Voight(2021), sec. 11.2.4, p. 168].

Remark 3.6 (concrete and abstract factor models)
The subgroups $Q, C \leq \mathbb{H}^\times$ are concrete: their elements are quaternions, and the Hurwitz action is quaternion conjugation. The abstract groups

$$Q_8, \qquad \mathbb{Z}/3\mathbb{Z} \tag{3.18}$$

remember the same group structures without remembering their quaternion coordinates.

Matching names and cardinalities does not identify these models. We need explicit isomorphisms

$$Q_8 \cong Q, \qquad \mathbb{Z}/3\mathbb{Z} \cong C. \tag{3.19}$$

The action must then be transported through these isomorphisms and proved equivariant. This keeps later abstract calculations connected to the quaternion realization that supplied them.

In Lean, `QuaternionGroup` at parameter 2 models the abstract quaternion factor, and `Multiplicative` applied to `ZMod` at modulus 3 models the abstract cyclic factor. The concrete subgroups remain the main computational model. Explicit isomorphisms recover the abstract interface from that model, and equivariance connects the transported action to quaternion conjugation. Formalization therefore keeps these bridges as results rather than treating them as definitions.

4 From the Hurwitz action to the binary tetrahedral group

The action calculated in the Hurwitz action on the quaternion group now lets us combine the quaternion group Q and the cyclic group C. We first recall the general steps: normalization makes a set product into a subgroup, complementarity gives unique factorization, and unique factorization produces an internal semidirect product.

Applying these steps inside $\mathbb{H}^\times$ gives a concrete group with 24 elements:

$$Q,\ C \longrightarrow T = Q \vee C = QC \longrightarrow T \cong Q \rtimes C. \tag{4.1}$$

The last isomorphism compares two models; it does not identify their elements.

Definition 4.1 (normalizer of a subgroup)

Let N be a subgroup of G. The **normalizer** of N in G is

$$N_G(N) = \{g \in G : gNg^{-1} = N\}. \tag{4.2}$$

It is the largest subgroup of G in which N is normal. In particular, a subgroup $H \le G$ acts on N by conjugation precisely when

$$H \le N_G(N). \tag{4.3}$$

See [Fré(2023), sec. 4.2.6, pp. 56–57].

The ambient group matters. If H normalizes N, then N is normal in the subgroup generated by N and H. This does not imply that N is normal in all of G.

For the quaternion subgroups in Quaternion and order-three subgroups, the three conjugation calculations in the Hurwitz action on the quaternion group give

$$C \le N_{\mathbb{H}^\times}(Q). \tag{4.4}$$

They do not claim that Q is normal in $\mathbb{H}^\times$.

Lemma 4.2 (a normalized product is the generated subgroup)
Let $N, H \le G$. If $H \le N_G(N)$, then the set product

$$NH = \{nh : n \in N,\ h \in H\} \tag{4.5}$$

is a subgroup and

$$N \vee H = NH. \tag{4.6}$$

Moreover, N is normal in $N \vee H$.

Proof. Normalization gives the calculation needed for closure:

$$(n_1h_1)(n_2h_2) = n_1(h_1n_2h_1^{-1})h_1h_2 \in NH. \tag{4.7}$$

It also gives $(nh)^{-1} = (h^{-1}n^{-1}h)h^{-1} \in NH$. Thus NH is a subgroup containing both N and H, while every subgroup containing N and H contains every product nh. This proves $N \vee H = NH$. Finally, conjugation by generators from N and H preserves N, so N is normal in their generated subgroup. □

This is the small bridge between an action by conjugation and an internal product. Compare the normalizer and internal-product discussion in [Fré(2023), secs. 4.2.6 and 4.2.13, pp. 56–57 and 62–63].

Definition 4.3 (complementary subgroups)

Subgroups $N, H \le G$ are **complementary**, in this ordered sense, when multiplication

$$N \times H \longrightarrow G, \qquad (n,h) \longmapsto nh \tag{4.8}$$

is a bijection. Equivalently, every $g \in G$ has a unique factorization $g = nh$.

For subgroups, the two conditions

$$N \cap H = \{1\}, \qquad NH = G \tag{4.9}$$

give complementarity. Existence follows from $NH = G$. For uniqueness, if $n_1h_1 = n_2h_2$, then

$$n_2^{-1}n_1 = h_2h_1^{-1} \in N \cap H, \tag{4.10}$$

so both sides are 1.

Trivial intersection alone is not enough: a complement must also provide the factorization of the ambient group. Fré uses these ingredients for an internal semidirect product in [Fré(2023), sec. 4.2.13, pp. 62–63].

Theorem 4.4 (the internal semidirect-product theorem)
Let $N, H \le G$. Suppose N is normal in G and N, H are complementary. Conjugation defines an action

$$\alpha : H \longrightarrow \mathrm{Aut}(N), \qquad \alpha(h)(n) = hnh^{-1}. \tag{4.11}$$

Give $N \times H$ the multiplication

$$(n_1,h_1)(n_2,h_2) = \big(n_1\alpha(h_1)(n_2), h_1h_2\big). \tag{4.12}$$

The resulting external semidirect product is denoted $N \rtimes_\alpha H$. Multiplication induces an isomorphism

$$N \rtimes_\alpha H \xrightarrow{\cong} G, \qquad (n,h) \longmapsto nh. \tag{4.13}$$

Proof. The displayed multiplication is chosen so that

$$(n_1h_1)(n_2h_2) = n_1(h_1n_2h_1^{-1})h_1h_2. \tag{4.14}$$

Thus the multiplication map preserves products. Complementarity makes it bijective, so it is an isomorphism. □

Conversely, the canonical copies of N and H in $N \rtimes_\alpha H$ are complementary, and the copy of N is normal. The external multiplication and its split exact sequence are developed in [Isaev and Rubakov(2018), sec. 1.4.2, pp. 58–61]; compare [Fré(2023), sec. 4.2.13, pp. 62–63]. An arbitrary extension by a normal subgroup need not split: the complement is extra structure.

Example 4.5 (the binary tetrahedral subgroup of the quaternions)
Inside $\mathbb{H}^\times$, let

$$T = Q \vee C. \tag{4.15}$$

The Hurwitz action in the Hurwitz action on the quaternion group says that C normalizes Q. Hence a normalized product is the generated subgroup gives

$$T = QC, \qquad Q \trianglelefteq T. \tag{4.16}$$

The groups Q and C have orders 8 and 3. The order of their intersection divides both, so

$$Q \cap C = \{1\}. \tag{4.17}$$

They are therefore complementary in T. Every element has a unique normal form

$$q\omega^r, \qquad q \in Q, \quad 0 \leq r < 3, \tag{4.18}$$

and the internal semidirect-product theorem gives

$$Q \rtimes_\alpha C \cong T, \qquad \alpha(c)(q) = cqc^{-1}. \tag{4.19}$$

The action orientation is fixed by the quaternion calculation:

$$i \longmapsto k \longmapsto j \longmapsto i. \tag{4.20}$$

Voight identifies this semidirect product with the Hurwitz unit group and the binary tetrahedral group in [Voight(2021), sec. 11.2.4, p. 168].

Example 4.6 (the twenty-four Hurwitz units)
The unique normal forms divide T into three disjoint rows:

(4.21)

Q	$Q\omega$	$Q\omega^2$
8 elements	8 elements	8 elements.

Therefore $|T| = 8 \cdot 3 = 24$.

The first row is

$$Q = \{\pm 1, \pm i, \pm j, \pm k\}. \tag{4.22}$$

Since

$$\omega = \frac{-1+i+j+k}{2}, \qquad \omega^2 = \frac{-1-i-j-k}{2}, \tag{4.23}$$

left multiplication by the 8 elements of Q gives

$$Q\omega \sqcup Q\omega^2 = \left\{ \frac{\epsilon_0 + \epsilon_1 i + \epsilon_2 j + \epsilon_3 k}{2} : \epsilon_r \in \{\pm 1\} \right\}. \tag{4.24}$$

The coordinates are distinct, so these are the 16 half-integral units. Together with Q, they are exactly the 24 Hurwitz units listed in [Voight(2021), sec. 11.2, p. 166]. Thus the subgroup T is the unit group of the Hurwitz order, not merely another group of the same cardinality.

Remark 4.7 (internal, external, and abstract models)
Three related objects have appeared. The internal model $T \le \mathbb{H}^\times$ has quaternions as its elements. The external semidirect product $Q \rtimes_\alpha C$ has pairs (q, c) as its elements. An abstract binary tetrahedral group can instead be specified without quaternion coordinates.

The isomorphism $(q, c) \mapsto qc$ compares the first two models. It is not an equality of their elements.

The next note transports the distinguished central involution and cardinality to an abstract binary tetrahedral model. Transporting the full factorization to factors such as `QuaternionGroup 2` and `Multiplicative (ZMod 3)` also needs explicit isomorphisms and equivariance proofs. None of this follows from the order computation $|T| = 24$ alone.

Definition 4.8 (the distinguished involution of abstract 2T)
Let $T \le \mathbb{H}^\times$ be the binary tetrahedral group of Hurwitz units from the binary tetrahedral subgroup of the quaternions, and let B be the abstract semidirect-product model obtained from the same Hurwitz action. Choose the group isomorphism

$$\rho : B \cong T \tag{4.25}$$

that compares these two models. Voight identifies T with the Hurwitz unit group and with $Q_8 \rtimes \mathbb{Z}/3\mathbb{Z}$ in [Voight(2021), sec. 11.2.4, p. 168].

The quaternion -1 belongs to T; it is central, its square is 1, and it is not 1. Define the **distinguished central involution** of B by

$$z = \rho^{-1}(-1). \tag{4.26}$$

Since an isomorphism preserves multiplication and reflects equality,

$$z \in Z(B), \qquad z^2 = 1, \qquad z \neq 1. \tag{4.27}$$

Thus the abstract group retains the central involution supplied by its quaternionic realization. The element z is not identified with the quaternion -1; the isomorphism ρ relates them.

The same isomorphism transports cardinality. Since the calculation in the twenty-four Hurwitz units gives $|T| = 24$,

$$|B| = |T| = 24. \tag{4.28}$$

The corresponding Lean theorem uses Mathlib's `Nat.card`; its statement is `Nat.card BinaryTetrahedral.Abstract = 24`.

The occurrence of -1 among the 24 Hurwitz units is explicit in [Voight(2021), sec. 11.2, p. 166].

5 Central involutions and complementary summands

A central element whose square is one produces two complementary idempotents whenever two is invertible. The construction separates a module into the two eigenspaces of that element and separates the algebra itself into two algebra factors. No semisimplicity or representation classification is needed.

The reading order follows the implications

$$\begin{array}{ccccc} z^2 = 1,\ z \in Z(A) & \longrightarrow & e_+, e_- & \longrightarrow & M = e_+M \oplus e_-M \\ & & \downarrow & & \\ & & A \cong e_+A \times e_-A. & & \end{array} \tag{5.1}$$

Centrality is needed for the lower algebra statement. The upper module decomposition is an additive direct sum from the complementary-idempotent equations; centrality makes its summands A-submodules.

Definition 5.1 (central involution in an algebra)
Let R be a commutative ring, let A be an associative unital R-algebra, and suppose $2 \cdot 1_R$ is invertible. Write

$$h = (2 \cdot 1_R)^{-1}. \tag{5.2}$$

A **central involution** for this construction is an element $z \in A$ such that

$$z^2 = 1_A, \qquad za = az \quad \text{for every } a \in A. \tag{5.3}$$

We then define

$$e_+ = h(1_A + z), \qquad e_- = h(1_A - z), \tag{5.4}$$

using the structural map $R \to A$ for the scalar h.

Only the equation $z^2 = 1_A$ is used. The element need not have exact order 2: the extra condition $z \neq 1_A$ merely excludes the degenerate case $e_- = 0$. The algebra A need not be commutative.

Example 5.2 (a central group element inside a Group Algebra)
Let G be a group and let $z_G \in G$ satisfy

$$z_G^2 = 1_G, \qquad z_G g = g z_G \quad \text{for every } g \in G. \tag{5.5}$$

In the Group Algebra $R[G]$, write $[z_G]$ for the basis element indexed by z_G. Then

$$[z_G]^2 = [1_G] = 1_{R[G]}, \tag{5.6}$$

and $[z_G]$ commutes with every basis element $[g]$. By linearity it is a **central involution** in the algebra. The Group Algebra and the passage from group representations to its modules are developed in [Sengupta(2010), secs. 3.1–3.3, pp. 39–42]; compare [Webb(2007), pp. 1–4].

The brackets matter. Assume R is nontrivial. Then $[z_G] \neq -1_{R[G]}$: their supports differ when $z_G \neq 1_G$; when $z_G = 1_G$, this is $1_{R[G]} \neq -1_{R[G]}$, since 2 is invertible.

Lemma 5.3 (the two idempotents of a central involution)
For the elements e_+ and e_- associated with a **central involution**,

$$e_+^2 = e_+, \qquad e_-^2 = e_-, \qquad e_+e_- = e_-e_+ = 0, \qquad e_+ + e_- = 1_A. \tag{5.7}$$

Both idempotents are central.

Proof. First,

$$e_+ + e_- = h(1+z) + h(1-z) = 2h = 1_A, \tag{5.8}$$

so $e_- = 1_A - e_+$. Since $2h = 1_R$ and $z^2 = 1_A$,

$$e_+^2 = h^2(1 + 2z + z^2) = 2h^2(1+z) = e_+. \tag{5.9}$$

The remaining identities now follow from this one idempotence calculation:

$$\begin{aligned} e_-^2 &= (1_A - e_+)^2 = 1_A - e_+ = e_-, \\ e_+e_- &= e_+(1_A - e_+) = 0, \\ e_-e_+ &= (1_A - e_+)e_+ = 0. \end{aligned} \tag{5.10}$$

Finally, each $e_\pm$ is a scalar linear combination of the central elements 1_A and z, so it is central. ∎

These equations are the elementary two-idempotent instance of the projection calculus discussed in [Sengupta(2010), sec. 4.5, pp. 63–68].

Example 5.4 (the central split of the real Group Algebra of 2T)
Let B be the abstract binary tetrahedral group, let $z \in B$ be its **distinguished central involution**, and put

$$A = \mathbb{R}[B]. \tag{5.11}$$

The group element z, its basis image $[z] \in A$, and the scalar -1_A are different kinds of objects. The image $[z]$ is central in A and satisfies $[z]^2 = 1_A$, by a central group element inside a Group Algebra.

The two elements

$$e_+ = \frac{1_A + [z]}{2}, \qquad e_- = \frac{1_A - [z]}{2} \tag{5.12}$$

therefore satisfy

$$e_+^2 = e_+, \qquad e_-^2 = e_-, \qquad e_+e_- = e_-e_+ = 0, \qquad e_+ + e_- = 1_A. \tag{5.13}$$

They are central by the two idempotents of a central involution. Hence left multiplication by e_+ and e_- gives two complementary projections on the regular A-module. This specializes the generic central-involution split to $\mathbb{R}[B]$; no representation-theoretic classification is used.

The Group Algebra convention is developed in [Sengupta(2010), secs. 3.1–3.3, pp. 39–42]. The projection interpretation of complementary idempotents is compared with [Webb(2007), exercise 2.7, p. 15].

Theorem 5.5 (the eigenspace decomposition of a module)
Let z be a **central involution** in A, let e_+, e_- be its associated idempotents, and let M be a left A-module. Left multiplication defines A-linear projections

$$p_\pm : M \longrightarrow M, \qquad p_\pm(m) = e_\pm m. \tag{5.14}$$

They satisfy

$$p_\pm^2 = p_\pm, \qquad p_+ + p_- = \mathrm{id}_M, \qquad p_+p_- = p_-p_+ = 0. \tag{5.15}$$

Their ranges and kernels are

$$\begin{aligned} \operatorname{range}(p_+) &= e_+M, \quad \ker(p_+) = \operatorname{range}(p_-) = e_-M, \\ \operatorname{range}(p_-) &= e_-M, \quad \ker(p_-) = \operatorname{range}(p_+) = e_+M. \end{aligned} \tag{5.16}$$

Consequently, if

$$M_+ = e_+M, \qquad M_- = e_-M, \tag{5.17}$$

then

$$M = M_+ \oplus M_-. \tag{5.18}$$

These summands are precisely the two eigenspaces for the action of z:

$$M_+ = \{m \in M : zm = m\}, \qquad M_- = \{m \in M : zm = -m\}. \tag{5.19}$$

Proof. The idempotent and mixed-product identities from the two idempotents of a central involution give the displayed projection identities after acting on M. Centrality of $e_\pm$ makes $p_\pm$ A-linear. Their ranges are $e_\pm M$ by definition. If $p_+(m) = 0$, then

$$m = (p_+ + p_-)(m) = p_-(m), \tag{5.20}$$

so m lies in the range of p_-. Conversely, $p_+p_- = 0$ shows that every element in that range lies in $\ker(p_+)$. This proves the first kernel-range equality; the second is symmetric. The identity $p_+ + p_- = \mathrm{id}_M$ now gives the displayed direct sum.

Direct calculation gives $ze_+ = e_+$ and $ze_- = -e_-$, proving one eigenspace inclusion in each case. Conversely, if $zm = m$, then

$$e_+m = h(m + zm) = 2hm = m. \tag{5.21}$$

If $zm = -m$, the analogous calculation gives $e_-m = m$. □

Idempotents as projections and complementary orthogonal idempotents as direct decompositions are recorded in [Webb(2007), exercise 2.7, p. 15]; see also [Sengupta(2010), props. 4.5.1–4.5.3, pp. 63–67].

Theorem 5.6 (the central-idempotent algebra product)
For a **central involution** z and its associated idempotents, put

$$A_+ = e_+A, \qquad A_- = e_-A. \tag{5.22}$$

Because e_+ and e_- are central, these are two-sided ideals and $e_\pm A = Ae_\pm$. Each is a unital R-algebra in its own right, with unit $e_\pm$ and structural map $r \mapsto re_\pm$. There is an R-algebra isomorphism

$$\begin{aligned} \Phi : A &\longrightarrow A_+ \times A_-, & a &\longmapsto (e_+a, e_-a), \\ \Psi : A_+ \times A_- &\longrightarrow A, & (x, y) &\longmapsto x + y. \end{aligned} \tag{5.23}$$

Proof. For $a, b \in A$, $a(e_\pm b) = e_\pm(ab)$ and $(e_\pm b)a = e_\pm(ba)$, so $A_\pm$ are two-sided ideals. If $x = e_+a \in A_+$, then $e_+x = x$; thus e_+ is the unit of A_+, and similarly for A_-.

Mixed products vanish:

$$(e_+a)(e_-b) = e_+e_-ab = 0, \tag{5.24}$$

and likewise in the reverse order. It follows that Φ and Ψ preserve multiplication. Finally,

$$\Psi\Phi(a) = (e_+ + e_-)a = a, \tag{5.25}$$

while $\Phi\Psi(x, y) = (x, y)$ because the matching idempotent fixes each factor and the other annihilates it. □

The use of local units on idempotent-generated ideals follows the standard discussion in [Sengupta(2010), secs. 4.5–4.6, pp. 63–73].

Example 5.7 (splitting the Group Algebra of C_2)
Let $C_2 = \{1, s\}$ with $s^2 = 1$, and suppose 2 is invertible in the nonzero commutative ring R. The basis element $[s]$ is central, and

$$e_+ = \frac{[1] + [s]}{2}, \qquad e_- = \frac{[1] - [s]}{2}. \tag{5.26}$$

The algebra-product theorem becomes

$$R[C_2] \cong R \times R. \tag{5.27}$$

The isomorphism and its inverse are explicit:

$$\begin{aligned} a[1] + b[s] &\longmapsto (a+b, a-b), \\ (x, y) &\longmapsto \frac{x+y}{2}[1] + \frac{x-y}{2}[s]. \end{aligned} \tag{5.28}$$

Under this map, e_+ goes to $(1, 0)$ and e_- goes to $(0, 1)$. Thus the two idempotents are the coordinate projections, not merely a dimension count.

Remark 5.8 (three levels of splitting)
The same formulas support three conclusions, but their structures should not be conflated:

level	conclusion	input
additive	$M = e_+M \oplus e_-M$	$e_+ + e_- = 1,\ e_+e_- = e_-e_+ = 0$
module	$M_\pm$ are eigensubmodules	central action of z
algebra	$A \cong e_+A \times e_-A$	e_+, e_- central.

(5.29)

For an arbitrary idempotent $e \in A$, left multiplication by e and $1 - e$ gives a split of the underlying R-module,

$$A = eA \oplus (1-e)A, \tag{5.30}$$

and therefore also of the underlying additive group. Without centrality, these projections need not be homomorphisms of the left regular A-module: eA and $(1-e)A$ are right ideals, but need not be left ideals or two-sided ideals. The two summands therefore need not be algebra factors. The product theorem is stronger than this R-module direct-sum statement.

Remark 5.9 (mathematical summands and physical readings)
A central involution canonically separates the algebra and its modules into two mathematical summands. This alone does not identify either summand with particles, interactions, chirality, or any other physical sector. Such an interpretation requires additional data: a representation, selected observables or forms, dynamics, and a map from the mathematics to measurable quantities.

The split is useful precisely because it can be calculated before those choices are made. Later notes may test proposed physical correspondences against it, but the algebra decomposition remains valid independently of whether any such proposal succeeds.

6 Real blocks of the quaternion Group Algebra

The preceding central-involution construction isolates algebra factors without classifying all representations. For the quaternion group, the following direct coefficient calculation goes further and makes every real block explicit.

Proposition 6.1 (the real Group Algebra of the quaternion group)
Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the concrete quaternion group from the quaternion group inside Hamilton's Quaternions, and put

$$\mathcal{E} = \{-1, 1\} \subset \mathbb{R}^\times, \qquad \mathcal{X} = \mathcal{E} \times \mathcal{E}. \tag{6.1}$$

For $(\epsilon, \delta) \in \mathcal{X}$, there is a real character

$$\chi_{\epsilon,\delta} : Q_8 \longrightarrow \mathbb{R}^\times \tag{6.2}$$

determined by

$$\chi_{\epsilon,\delta}(-1) = 1, \qquad \chi_{\epsilon,\delta}(i) = \epsilon, \qquad \chi_{\epsilon,\delta}(j) = \delta, \qquad \chi_{\epsilon,\delta}(k) = \epsilon\delta. \tag{6.3}$$

These are the four characters pulled back from $Q_8/\{\pm 1\} \cong C_2 \times C_2$; compare [Etingof et al.(2011), sec. 4.3, pp. 63–64 and ex. 4.8.1, p. 73].

Write $\mathbb{R}^{\mathcal{X}}$ for the algebra of functions $\mathcal{X} \to \mathbb{R}$, with pointwise operations. Extending the four characters and the inclusion $Q_8 \subset \mathbb{H}^\times$ linearly gives an algebra homomorphism

$$\Phi : \mathbb{R}[Q_8] \longrightarrow \mathbb{R}^{\mathcal{X}} \times \mathbb{H}. \tag{6.4}$$

Then Φ is an isomorphism of real algebras. The displayed map retains the chosen character index and quaternionic realization as part of its data.

Proof. Let u_q be the Group-Algebra basis element indexed by $q \in Q_8$.

Write a general element in the form

$$x = \sum_{q \in Q_8} a_q u_q. \tag{6.5}$$

For $q \in \{1, i, j, k\}$, set

$$s_q = a_q + a_{-q}, \qquad d_q = a_q - a_{-q}. \tag{6.6}$$

The quaternion coordinate of $\Phi(x)$ is

$$d_1 + d_i i + d_j j + d_k k. \tag{6.7}$$

It therefore recovers the four differences d_q.

The coordinate indexed by $(\epsilon, \delta) \in \mathcal{X}$ is

$$y_{\epsilon,\delta} = s_1 + \epsilon s_i + \delta s_j + \epsilon\delta s_k. \tag{6.8}$$

The inverse Hadamard calculation gives

$$\begin{aligned} s_1 &= \frac{1}{4} \sum_{(\epsilon,\delta)\in\mathcal{X}} y_{\epsilon,\delta}, \quad & s_i &= \frac{1}{4} \sum_{(\epsilon,\delta)\in\mathcal{X}} \epsilon y_{\epsilon,\delta}, \\ s_j &= \frac{1}{4} \sum_{(\epsilon,\delta)\in\mathcal{X}} \delta y_{\epsilon,\delta}, \quad & s_k &= \frac{1}{4} \sum_{(\epsilon,\delta)\in\mathcal{X}} \epsilon\delta y_{\epsilon,\delta}. \end{aligned} \tag{6.9}$$

Thus the four real coordinates recover the four sums s_q. Finally,

$$a_q = \frac{s_q + d_q}{2}, \qquad a_{-q} = \frac{s_q - d_q}{2}, \tag{6.10}$$

so $\Phi(x)$ recovers all eight coefficients of x. Hence Φ is injective. Its domain and codomain both have real dimension 8, so it is bijective. $\square$

The quaternion coordinate alone is surjective, because its basis values include $1, i, j, k$. It vanishes exactly when $d_1 = d_i = d_j = d_k = 0$, or equivalently when $a_q = a_{-q}$ for $q \in \{1, i, j, k\}$. Its kernel is therefore the 4-dimensional span of

$$u_1 + u_{-1}, \quad u_i + u_{-i}, \quad u_j + u_{-j}, \quad u_k + u_{-k}. \tag{6.11}$$

It therefore induces

$$\mathbb{R}[Q_8]/\ker(\Phi_{\mathbb{H}}) \cong_{\mathbb{R}\text{-alg}} \mathbb{H}. \tag{6.12}$$

This quotient depends on the chosen quaternionic realization. The combined map Φ is what exhibits the selected quotient as the quaternion factor of the displayed product; a quotient map by itself does not supply that direct-factor statement.

The quaternion model is grounded in [Voight(2021), sec. 11.2, p. 166]. The extension of group representations to Group-Algebra maps follows [Sengupta(2010), secs. 3.1–3.2, pp. 39–41]. The coefficient recovery above is direct and does not assume a general classification theorem. No physical meaning is assigned to a factor merely from its dimension or familiar algebra.

7 Appendix

The appendix first supplies the general Group-Algebra route behind the finite examples. It then records exploratory structures that are not needed for the quaternionic and Group-Algebra spine.

7.1 The human-curated Group-Algebra route

A finite group becomes linear once its elements are used as a basis. This route starts with that construction, passes through modules, semisimple blocks, and characters, and then follows normal subgroups, computation, and twisted multiplication. The explicit quaternion calculations earlier in these notes are test cases for this general route, not substitutes for it.

Definition 7.1.1 (the Group Algebra and its universal extension `MonoidAlgebra.lift` `MonoidAlgebra.lift_single`)
Let k be a commutative ring and G a group. The **Group Algebra** $k[G]$ is the free k-module with basis $[g]$ indexed by $g \in G$, equipped with

$$[g][h] = [gh], \qquad 1 = [1_G], \qquad \Big(\sum_g a_g[g]\Big)\Big(\sum_h b_h[h]\Big) = \sum_{g,h} a_g b_h [gh]. \tag{7.1.1}$$

The scalar embedding sends a to $a[1_G]$. Thus each group element is a unit of $k[G]$, with inverse $[g^{-1}]$.

If A is a k-algebra and $u : G \to A^\times$ is a group homomorphism, there is a unique k-algebra homomorphism

$$\widetilde{u} : k[G] \longrightarrow A, \qquad \widetilde{u}\left(\sum_g a_g[g]\right) = \sum_g a_g u(g). \tag{7.1.2}$$

This universal property extends representations and concrete group maps from basis elements to the whole Group Algebra. See [Sengupta(2010), secs. 3.1–3.2, pp. 39–41].

Theorem 7.1.2 (representations are Group-Algebra modules `Representation.asModule` `Representation.ofModule` `Rep.equivalenceModuleMonoidAlgebra`)
Let k be a field, G a group, and V a k-vector space. Giving a linear representation $\rho : G \to \mathrm{GL}_k(V)$ is equivalent to giving a unital left $k[G]$-module structure on V, up to the identity type synonyms used to keep the

2 scalar actions distinct. The action associated to ρ is

$$(\sum_g a_g[g]) \cdot v = \sum_g a_g \rho(g) v. \tag{7.1.3}$$

Conversely, restriction of a unital $k[G]$-action to the units $[g]$ recovers ρ. Invariant subspaces are submodules and intertwining linear maps are $k[G]$-linear maps.

Proof. The extension in the Group Algebra and its universal extension supplies the algebra homomorphism $k[G] \to \mathrm{End}_k(V)$. Associativity and the unit law give a module. In the other direction, multiplication by each $[g]$ is invertible, and the module law gives the group law. These operations are inverse. See [Sengupta(2010), sec. 3.2, pp. 40–41]. □

Example 7.1.3 (the regular representation reads the basis)
Let G act on $k[G]$ by left multiplication. In the basis $\{[h] : h \in G\}$, an element g sends $[h]$ to $[gh]$; hence its matrix is the permutation matrix of left translation. This is the **left regular representation**. It is faithful because the image of $[1_G]$ records g.

The same basis underlies the coefficient calculations in the real Group Algebra of the quaternion group: there, the quaternion coordinate recovers antisymmetric coefficient pairs and the character coordinates recover symmetric pairs. The regular module is the linear carrier for both the abstract Group Algebra and the explicit 8-coordinate computation. Compare [James and Liebeck(2001), ch. 6, pp. 53–58].

Theorem 7.1.4 (Maschke averaging `MonoidAlgebra.Submodule.exists_isCompl`)
Let G be finite and k a field in which $|G|$ is invertible. Every subrepresentation $W \subseteq V$ has a G-stable complement. Consequently every finite-dimensional $k[G]$-module, and $k[G]$ itself, is semisimple.

Proof. Choose a k-linear projection $P : V \to W$ and average its conjugates:

$$P_G = \frac{1}{|G|} \sum_{g \in G} \rho(g) P \rho(g)^{-1}. \tag{7.1.4}$$

Then P_G remains the identity on W and commutes with G. Its kernel is a G-stable complement. Division by $|G|$ requires the stated invertibility hypothesis. This is Maschke's theorem as proved in [Sengupta(2010), thm. 3.5.1, pp. 44–46]. □

Theorem 7.1.5 (Wedderburn blocks over an algebraically closed field `exists_algEquiv_pi_matrix_conjClasses`)
Let G be finite and k an algebraically closed field whose characteristic does not divide $|G|$. There are positive integers d_C, indexed by the conjugacy classes C of G, such that

$$k[G] \cong_{k\text{-alg}} \prod_{C \in \mathrm{Conj}(G)} M_{d_C}(k), \qquad \sum_C d_C^2 = |G|. \tag{7.1.5}$$

The class labels are an indexing choice: the statement does not canonically pair a particular conjugacy class with a particular irreducible module.

Maschke semisimplicity followed by Artin–Wedderburn gives the stated product. Lux and Pahlings state the results in [Lux and Pahlings(2010), thms. 1.5.5–1.5.6, pp. 56–57].

The dimension identity can also be read from the regular representation, where an irreducible module occurs with multiplicity equal to its dimension; compare [James and Liebeck(2001), thms. 11.9 and 11.12, pp. 100–101].

Definition 7.1.6 (characters, class functions, and the character table `basisOfIrreducibleCharacters` `characterTable` `card_inv_mul_sum_characterTable_mul_characterTable_inv` `card_conjClass_mul_sum_characterTable_mul_characterTable_inv`)
For a finite-dimensional representation ρ over a characteristic-zero field, its character is $\chi_\rho(g) = \mathrm{tr}(\rho(g))$. Trace is unchanged by conjugation, so χ_ρ is a class function. A **character table** places the irreducible characters in rows and the conjugacy classes in columns. Tau Ceti identifies class functions with functions on conjugacy classes in `TauCeti.ClassFunction.equivConjClasses`.

Over $\mathbb{C}$, the irreducible characters form an orthonormal basis of the class functions. Thus the table is square: the number of irreducible characters equals the number of conjugacy classes. This also counts the Wedderburn blocks and gives the dimension of the center. See [James and Liebeck(2001), thm. 16.4, pp. 159–166].

Theorem 7.1.7 (primitive central idempotents as block projectors `primitiveCentralIdempotent` `asAlgebraHom_primitiveCentralIdempotent` `primitiveCentralIdempotent_mul_primitiveCentralIdempotent`)

Over a splitting field of characteristic not dividing $|G|$, an irreducible character χ defines the central element

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})[g]. \tag{7.1.6}$$

It is a nonzero primitive central idempotent. On a simple module with character ψ, it acts as the identity if $\psi = \chi$ and as zero otherwise. Distinct e_χ are therefore orthogonal block projectors. See [Lux and Pahlings(2010), thm. 2.1.6 and cor. 2.1.7, pp. 88–90]. A constructed family of such projectors does not yield a complete block decomposition until a sum-to-1 statement is also known.

Theorem 7.1.8 (the Frobenius–Schur trichotomy `frobeniusSchurIndicator_eq_one_or_eq_zero_or_eq_neg_one` `frobeniusSchurIndicator_eq_one_iff` `frobeniusSchurIndicator_eq_neg_one_iff` `frobeniusSchurIndicator_eq_zero_iff`)

Let ρ be an irreducible finite-dimensional representation of a finite group over an algebraically closed field of characteristic 0. Its Frobenius–Schur indicator

$$\nu_2(\rho) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g^2) \tag{7.1.7}$$

takes exactly 1 of the values 1, 0, and -1. The value 1 is equivalent to the existence of a nondegenerate invariant symmetric bilinear form; -1 is equivalent to a nondegenerate invariant alternating form; and 0 is equivalent to the absence of a nonzero invariant bilinear form. See [James and Liebeck(2001), def. 23.13, the proof of thm. 23.14, and thm. 23.16, pp. 273–277]. James and Liebeck state the alternatives using nonzero invariant forms; for an irreducible representation, a nonzero invariant symmetric or alternating form has zero radical and is therefore nondegenerate.

This trichotomy classifies the invariant-form behavior of the complex irreducible representation. It does not by itself compute a Schur index or a complete decomposition after descent to $\mathbb{R}$.

Remark 7.1.9 (real types are not Schur indices)
Over $\mathbb{R}$, a simple finite-dimensional block may instead have division algebra $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The Frobenius–Schur trichotomy in the Frobenius–Schur trichotomy detects invariant-form type, but it is not a complete theorem about a character's field of values or Schur index. The explicit decomposition

$$\mathbb{R}[Q_8] \cong \mathbb{R}^4 \times \mathbb{H} \tag{7.1.8}$$

in the real Group Algebra of the quaternion group is a complete real example, not an application of the algebraically closed formula. For Schur indices and invariant forms, see [Lux and Pahlings(2010), sec. 2.9, pp. 164–172].

Definition 7.1.10 (conjugate constituents and the inertia subgroup)
Let $N \trianglelefteq G$, let V be a G-representation, and let $W \subseteq V|_N$ be an irreducible N-subrepresentation. For $g \in G$, the **conjugate representation** ${}^{g}W$ has the same vector space and action

$$n \cdot_g w = (g^{-1}ng) \cdot w. \tag{7.1.9}$$

The **inertia subgroup** of W is

$$I_G(W) = \{g \in G : {}^{g}W \cong W\}. \tag{7.1.10}$$

The isomorphism classes of conjugates form a G-orbit, with stabilizer $I_G(W)$. See [Lux and Pahlings(2010), sec. 3.6, pp. 222–225].

Theorem 7.1.11 (the single orbit in Clifford restriction `isSemisimpleRepresentation_comp_subtype` `exists_isAtom_forall_nonempty_linearEquiv_conjSubrep`)
Let $N \trianglelefteq G$ be finite, and work over a field for which the relevant finite-group modules are semisimple. If V is an irreducible G-representation, then the irreducible constituents of $V|_N$ lie in 1 G-orbit under conjugation. Equivalently, after choosing 1 constituent W, every other constituent is isomorphic to some ${}^{g}W$.

This is the orbit assertion in Clifford theory. It does not by itself say that all constituents occur with the same multiplicity, nor does it construct extensions across the inertia subgroup. Compare [Lux and Pahlings(2010), thm. 3.6.2(a), pp. 223–225].

Theorem 7.1.12 (Clifford homogeneity and the semidirect route)
Let $N \trianglelefteq G$ and let V be an irreducible complex representation of G. If $W_1, \ldots, W_r$ represent the distinct conjugates of an irreducible constituent of $V|_N$, then for 1 positive integer e,

$$V|_N \cong e(W_1 \oplus \cdots \oplus W_r). \tag{7.1.11}$$

Thus restriction is homogeneous along a single orbit. The common multiplicity is an additional assertion beyond the single orbit in Clifford restriction. This is Clifford's theorem, [James and Liebeck(2001), thm. 20.8, pp. 216–217].

The inertia subgroup is the intermediate group where one studies whether W extends, possibly projectively, before inducing to G. For semidirect products, this reduces the problem to the action of the complement. For $Q_8 \rtimes C_3 \cong 2T$, it connects the normal quaternion subgroup to the skew-Group-Algebra model in two routes from binary tetrahedral symmetry. This theorem does not supply the full character table of $2T$.

Example 7.1.13 (semidirect products make Clifford theory concrete)
Suppose $G = N \rtimes H$. Conjugation by H permutes the irreducible constituents of $V|_N$. Their stabilizers are the inertia subgroups, so the orbit and homogeneity statements in the single orbit in Clifford restriction and Clifford homogeneity and the semidirect route reduce the search for irreducible G-representations to stabilizer data, projective extension, and induction. Clifford theory thereby gives a character-theoretic route for semidirect products.

For $Q_8 \rtimes C_3 \cong 2T$, the action and ordinary skew-Group-Algebra realization are recorded in the central split of the real Group Algebra of 2T and two routes from binary tetrahedral symmetry. They supply the normal-subgroup and algebraic input to this route, but do not give the full character table of $2T$.

Remark 7.1.14 (from a presentation to checked representation data)
A finite presentation $\langle x_1, \ldots, x_m \mid r_1, \ldots, r_s \rangle$ gives a compact input for algorithms that enumerate cosets, compute conjugacy classes, and build character data. Sims develops the algorithms and the conditions under which such computations terminate; Lux and Pahlings place them inside computational representation theory.

The output has 3 possible evidential strengths. A transcript with versioned inputs is reproducible. A compact certificate, such as matrices satisfying the relations together with independently checked completeness identities, can be verified without trusting the search. An unrecorded software answer is neither. GAP is therefore a discovery and calculation tool; its output becomes mathematics here only when the decisive relations and completeness checks are visible. See [Sims(1994), ch. 1] and [Lux and Pahlings(2010), secs. 1.1 and 4.2].

The route is

$$\begin{aligned} \text{finite presentation} &\longrightarrow \text{computed candidates} \\ &\longrightarrow \text{checked relations and completeness.} \end{aligned} \tag{7.1.12}$$

The arrows distinguish generation from certification rather than describing a project workflow.

Definition 7.1.15 (factor sets and twisted Group Algebras `of_mul_of` `lift` `isProjectiveRepEquivAlgHom`)
Let k be a field, G a group, and $\alpha : G \times G \to k^\times$ a normalized 2-cocycle:

$$\alpha(1, g) = \alpha(g, 1) = 1, \qquad \alpha(g, h)\alpha(gh, \ell) = \alpha(h, \ell)\alpha(g, h\ell). \tag{7.1.13}$$

The **twisted Group Algebra** $k_\alpha[G]$ has basis u_g and multiplication

$$u_g u_h = \alpha(g, h) u_{gh}. \tag{7.1.14}$$

The cocycle equation is exactly the associativity condition.

A projective representation with factor set α, namely operators T_g satisfying $T_g T_h = \alpha(g, h) T_{gh}$, extends uniquely to an algebra map $k_\alpha[G] \to \mathrm{End}_k(V)$; conversely an algebra map gives such a projective representation on its basis operators. Cohomologous cocycles rescale the basis and give isomorphic twisted algebras. See [Conlon(1964), the introduction and sec. 1, pp. 152–155].

Remark 7.1.16 (the qualified Clifford bridge)
A generalized Clifford algebra becomes a twisted group algebra only after its parameters are fixed. Let k contain a primitive n-th root of unity ω, assume $\operatorname{char} k \nmid n$, and choose $q_1, \ldots, q_m \in k^\times$. The algebra generated by $e_1, \ldots, e_m$ with

$$e_i^n = q_i, \qquad e_i e_j = \omega e_j e_i \quad (j < i) \tag{7.1.15}$$

is isomorphic to a twisted group algebra of $(\mathbb{Z}/n\mathbb{Z})^m$ for an explicit 2-cocycle. The monomials in the e_i correspond to its twisted basis. See [Cheng et al.(2019), sec. 2.3 and prop. 2.1, pp. 3–4].

This is a qualified bridge, not an identification of every Clifford algebra with an ordinary Group Algebra. The field, root of unity, grading, parameters, and twisting are part of the theorem. It is also different from Clifford theory in the single orbit in Clifford restriction and Clifford homogeneity and the semidirect route, which concerns restriction of representations to normal subgroups.

Remark 7.1.17 (the inertia obstruction is projective)
For a constituent W with inertia subgroup $I = I_G(W)$, choose intertwiners $T_i : W \to {}^iW$. Their composites need not satisfy the group law strictly. Instead one obtains scalars

$$T_i T_j = \alpha(i, j) T_{ij}. \tag{7.1.16}$$

Associativity makes α a factor set. Changing the intertwiners changes α by a coboundary. The resulting cohomology class measures the obstruction to replacing the projective action by an ordinary action.

Twisted Group Algebras therefore enter the Clifford-theory route. The ordinary skew Group Algebra in two routes from binary tetrahedral symmetry corresponds to trivial twisting; it should not be identified with a general $k_\alpha[G]$. The extension and induction analysis is developed in [Lux and Pahlings(2010), secs. 3.6–3.7, pp. 222–240].

Proposition 7.1.18 (the canonical grading of a Group Algebra `AddMonoidAlgebra.grade.isInternal` `AddMonoidAlgebra.grade.gradedAlgebra` `AddMonoidAlgebra.single_mem_grade`)

For each $g \in G$, let $A_g = k[g]$ be the 1-dimensional subspace spanned by the basis element $[g]$. Then

$$k[G] = \bigoplus_{g \in G} A_g, \qquad A_g A_h \subseteq A_{gh}. \tag{7.1.17}$$

Thus $k[G]$ is canonically graded by the group G. In Lean's additive graded-algebra interface this statement is expressed after passing from the multiplicative index G to its additive copy; each single basis element lies in its corresponding grade.

The basis and multiplication law in the Group Algebra and its universal extension give the decomposition and product containment directly; compare the construction in [Sengupta(2010), secs. 3.1–3.2, pp. 39–41]. It is distinct from the Hopf structure in the standard Hopf structure on a Group Algebra: the grading records where products land, while the coalgebra maps record how a basis element is copied and inverted.

Proposition 7.1.19 (the standard Hopf structure on a Group Algebra `MonoidAlgebra.instHopfAlgebra` `MonoidAlgebra.counit_single` `MonoidAlgebra.comul_single` `MonoidAlgebra.antipode_single`)

Let k be a commutative ring and G a group. The Group Algebra $k[G]$ is a Hopf algebra with structure determined on basis elements by

$$\Delta([g]) = [g] \otimes [g], \qquad \epsilon([g]) = 1, \qquad S([g]) = [g^{-1}]. \tag{7.1.18}$$

The comultiplication and counit are algebra homomorphisms; the antipode extends inversion and satisfies the 2 convolution identities.

Proof. Each formula respects multiplication on basis elements. For example, $\Delta([gh]) = [gh] \otimes [gh] = ([g] \otimes [g])([h] \otimes [h])$. The counit calculation is similar. Finally $[g][g^{-1}] = [1] = [g^{-1}][g]$, which gives both antipode identities on the basis and hence by linearity on all of $k[G]$. This group-algebra Hopf structure appears, for example, in [Broué(2024), ex. 18.3.3(1), pp. 500–501]. □

Remark 7.1.20 (what the real quaternion example settles)
The general route has several distinct endpoints. Over an algebraically closed field, Wedderburn blocks and the character table organize all simple representations. Over $\mathbb{R}$, division-algebra type and descent data enter. For a normal subgroup, Clifford theory organizes restriction and induction. With a nontrivial factor set, projective representations are modules over a twisted algebra.

The quaternion calculation in the real Group Algebra of the quaternion group closes 1 bounded endpoint: it gives an explicit real-algebra equivalence $\mathbb{R}[Q_8] \cong \mathbb{R}^4 \times \mathbb{H}$, and Real blocks of the quaternion Group Algebra identifies its center. It does not prove a general real Wedderburn theorem, the characters of $2T$, or the Clifford-theory extension step. Those are separate statements and do not follow from the worked example.

Remark 7.1.21 (the Hessian-group outlook)
Wilson's proposed route begins with

$$Q_8 \rtimes C_3 \cong 2T \tag{7.1.19}$$

and then studies the iterated semidirect product $(G_{27} \rtimes Q_8) \rtimes C_3$, with the indicated actions part of the data. Its central scalar subgroup $C_3 \leq G_{27}$ has quotient the Hessian group of order 216, so the larger group is a triple cover of that quotient. The name alone would not choose either semidirect action. See [Wilson(2024), sec. 2.2, pp. 4–5].

The comparison with $U(1) \times SU(2) \times SU(3)$ is an attributed finite-model proposal, not an isomorphism or a canonical physical interpretation. To become a mathematical correspondence it would need typed representations and homomorphisms, the preserved forms or observables, and a rule connecting them to physical data. Hamilton's geometric-algebra models in [Hamilton(2023)] and [Hamilton and McMaken(2023)] provide a comparison motivation, not independent confirmation of this finite-group construction.

7.2 Finite tests for a topos layer

Group Actions, Group Algebras, and action groupoids already describe much of the finite symmetry data needed in these notes. The question here is narrower: does passing to a **topos** add universal classification, a common inverse-image language, or internal reasoning that is not already present below the topos level?

Five finite tests give a guarded answer. Ordinary actions and action groupoids do not need topoi. Torsor classification, inverse-image constructions, and internal inhabitation do add something, but only when their extra structure is actually used. Questions about local and global symmetry motivate these tests; they do not decide them.

Example 7.2.1 (the real Group Algebra of a cyclic group of order three)
Let $C_3 = \langle g \mid g^3 = 1 \rangle$. Sending g to x gives

$$\mathbb{R}[C_3] \cong \mathbb{R}[x]/(x^3 - 1). \tag{7.2.1}$$

Over the real numbers,

$$x^3 - 1 = (x - 1)(x^2 + x + 1), \tag{7.2.2}$$

and the two factors are coprime. The Chinese remainder theorem and the identification $\mathbb{R}[x]/(x^2 + x + 1) \cong \mathbb{C}$ therefore give

$$\mathbb{R}[C_3] \cong \mathbb{R} \times \mathbb{C}. \tag{7.2.3}$$

More concretely, if ζ is a primitive complex cube root of unity, the isomorphism is

$$a + bg + cg^2 \longmapsto \big(a + b + c,\ a + b\zeta + c\zeta^2\big). \tag{7.2.4}$$

The two factors can be read in three compatible ways:

factor of $x^3 - 1$	real algebra block	real representation
$x - 1$	$\mathbb{R}$	trivial line
$x^2 + x + 1$	$\mathbb{C}$	rotation plane.

(7.2.5)

The rotation plane is the linearization of the nontrivial part of the regular **Group Action**. More explicitly, the generator cyclically permutes the basis $1, g, g^2$. The averaging element

$$e_0 = \frac{1 + g + g^2}{3} \tag{7.2.6}$$

projects onto the fixed line $\mathbb{R}(1 + g + g^2)$, while its complementary kernel is the augmentation plane

$$\{a + bg + cg^2 : a + b + c = 0\}. \tag{7.2.7}$$

Orbits, freeness, and transitivity belong to the action; the projector and invariant subspaces appear only after linearization.

Since the spectrum of a product is the disjoint union of the spectra,

$$\operatorname{Spec}\mathbb{R}[C_3] \cong \operatorname{Spec}\mathbb{R} \sqcup \operatorname{Spec}\mathbb{C}. \tag{7.2.8}$$

This is a useful commutative picture. It is not yet a reason to introduce a topos, and it does not extend without choices to noncommutative Group Algebras.

Remark 7.2.2 (the center is a commutative shadow)
The quaternion group Q_8 has a real Group Algebra decomposition

$$\mathbb{R}[Q_8] \cong \mathbb{R}^4 \times \mathbb{H}. \tag{7.2.9}$$

This can be seen directly from the concrete copy $Q_8 \subset \mathbb{H}^\times$ in the quaternion group inside Hamilton's Quaternions, rather than assumed from a classification theorem.

Let u_q denote the basis element of the Group Algebra indexed by $q \in Q_8$, and write a general element as $x = \sum_{q \in Q_8} a_q u_q$. The four characters of $Q_8/\{\pm 1\} \cong C_2 \times C_2$ and the quaternionic map $u_q \mapsto q$ combine to an algebra homomorphism

$$\Phi : \mathbb{R}[Q_8] \longrightarrow \mathbb{R}^4 \times \mathbb{H}. \tag{7.2.10}$$

For $q \in \{1, i, j, k\}$, put $s_q = a_q + a_{-q}$ and $d_q = a_q - a_{-q}$. The quaternion coordinate of Φ recovers

$$d_1 + d_i i + d_j j + d_k k. \tag{7.2.11}$$

The four real coordinates recover the Hadamard transform

$$s_1 + \epsilon s_i + \delta s_j + \epsilon\delta s_k, \qquad \epsilon, \delta \in \{\pm 1\}. \tag{7.2.12}$$

The Hadamard matrix is invertible, so these four values recover every s_q. Together with the four differences, they recover every coefficient. Thus Φ has zero kernel. Both sides have real dimension 8, so Φ is an isomorphism. The quaternionic realization and multiplication used here are developed in Hamilton's Quaternions and [Voight(2021), sec. 11.2, p. 166].

Taking centers now gives

$$Z\big(\mathbb{R}[Q_8]\big) \cong Z\big(\mathbb{R}^4 \times \mathbb{H}\big) \cong \mathbb{R}^5. \tag{7.2.13}$$

The comparison is:

$$\begin{array}{c|c|c} \text{object} & \text{retained data} & \text{forgotten data} \\ \hline \mathbb{R}[Q_8] & 4\mathbb{R} \text{ and } \mathbb{H} & \text{none here} \\ Z(\mathbb{R}[Q_8]) & \text{five central factors} & \mathbb{H} \text{ and block size or type.} \end{array} \tag{7.2.14}$$

For a real-centered simple block, the center alone cannot distinguish matrix size or real from quaternionic type. A complex block still has complex center. Hence $\operatorname{Spec} Z(\mathbb{R}[Q_8])$ is a selected commutative shadow, not a lossless localization of the noncommutative algebra.

Example 7.2.3 (isotropy survives the coarse quotient)
Let $C_2 = \{1, s\}$ act on $X = \{-1, 0, 1\}$ by $s \cdot x = -x$. The action groupoid $C_2 \ltimes X$ has the points of X as objects and an arrow $(h, x) : x \longrightarrow h \cdot x$ for every $h \in C_2$. Omitting identity arrows, its shape is

$$\begin{tikzcd} -1 \arrow[r, bend left, "s"] & 1 \arrow[l, bend left, "s"] & & 0 \arrow[loop right, "s"] \end{tikzcd}$$

Thus

$$\operatorname{Aut}(0) \cong C_2, \qquad \operatorname{Aut}(1) = \operatorname{Aut}(-1) = 1. \tag{7.2.15}$$

The coarse quotient remembers only the two orbits:

$$\{-1, 1\} \qquad\qquad \{0\}.$$

It has forgotten the nonidentity automorphism at 0. By contrast, the action groupoid has one component equivalent to the terminal groupoid and one component equivalent to BC_2. Therefore

$$\mathsf{Set}^{(C_2 \ltimes X)^{\mathrm{op}}} \simeq \mathsf{Set} \times \mathsf{B}C_2, \tag{7.2.16}$$

while sheaves on the discrete coarse quotient form $\mathsf{Set} \times \mathsf{Set}$.

The retained information is already the isotropy arrow in the action groupoid. Passing to its **presheaf** category organizes families of such data, but does not create that arrow. This test supports the groupoid bridge and does not, by itself, admit a topos layer.

Definition 7.2.4 (torsors and the universal torsor)
Let G be a group object in a **topos** $\mathcal{E}$. A left G-object T is a **torsor** when $T \longrightarrow 1$ is

an epimorphism and

$$(\mu, \pi_2) : G \times T \longrightarrow T \times T, \qquad (g, t) \longmapsto (g \cdot t, t) \tag{7.2.17}$$

is an isomorphism. The second condition is the internal form of freeness and transitivity. See [Mac Lane and Moerdijk(1992), sec. VIII.2, pp. 429–430].

For an ordinary group G, write

$$\mathsf{BG} = \mathsf{Set}^{BG^{\mathrm{op}}} \tag{7.2.18}$$

for the topos of right G-sets, regarded as **presheaf**s on the one-object category BG. Its **universal torsor** U_G has underlying right G-set G with regular right multiplication. The constant group object $\underline{G}$, whose right G-action is trivial, acts on U_G by left multiplication. This supplies the torsor action; the left and right actions commute.

For C_2, the regular left action is a torsor in Set:

$$C_2 \times C_2 \longrightarrow C_2 \times C_2, \qquad (g, h) \longmapsto (gh, h) \tag{7.2.19}$$

is a bijection. The trivial action on a 2-point set D, despite $D \to 1$ being onto, is not a torsor. Its corresponding map sends (g, d) to (d, d), so it is neither injective nor surjective. Inhabitation and cardinality alone do not supply a torsor.

Theorem 7.2.5 (the classifying property of BG)
Let G be a group and $\mathcal{E}$ a topos over Set. There is a natural equivalence

$$\mathrm{Geom}_{/\mathsf{Set}}(\mathcal{E}, \mathsf{BG}) \simeq \mathrm{Tor}(\mathcal{E}, G) \tag{7.2.20}$$

between geometric morphisms over Set to BG and G-torsors in $\mathcal{E}$. Under this equivalence, a **geometric morphism**

$$f : \mathcal{E} \longrightarrow \mathsf{BG} \tag{7.2.21}$$

classifies the torsor f^*U_G.

This is Theorem VIII.2.7 of [Mac Lane and Moerdijk(1992)]; its theorem and proof are on printed pp. 431–433.

The two directions are visible in the diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, "f"] & \mathsf{BG} \\ f^*U_G & U_G \arrow[l, maps to, "f^*"'] \end{tikzcd}$$

The geometric morphism points toward the classifying topos. Its inverse-image functor carries the universal torsor back to the topos in which the classified torsor lives.

Proof. Inverse image preserves finite limits and, as a left adjoint, colimits. Every epimorphism in a topos is regular, so inverse image carries the universal torsor equations, including the epimorphic map to 1, to torsor equations in $\mathcal{E}$. Conversely, the cited theorem constructs a geometric morphism from a torsor and proves that the two constructions are naturally inverse. The complete construction belongs to the classifying theorem; the point used here is its variance and universal property. ☐

For a single finite action, freeness and transitivity can be checked without topoi. The added value here is one universal object classifying torsors in every topos over Set and carrying them along inverse image.

Convention 7.2.6 (geometric morphisms and inverse image)

For a **geometric morphism**

$$f : \mathcal{E} \longrightarrow \mathcal{F}, \tag{7.2.22}$$

we write

$$f^* : \mathcal{F} \longrightarrow \mathcal{E}, \qquad f_* : \mathcal{E} \longrightarrow \mathcal{F}, \qquad f^* \dashv f_*. \tag{7.2.23}$$

Thus the morphism and its inverse-image functor point in opposite directions. The functor f^* preserves finite limits.

Two examples fix the convention:

input map	geometric morphism	inverse image
$i : \{x\} \hookrightarrow X$	$\mathsf{Set} \to \mathsf{Sh}(X)$	$i^*F = F_x = \varinjlim_{x \in U} F(U)$
$\varphi : H \to G$	$\mathsf{B}H \to \mathsf{B}G$	$\mathrm{Res}^G_H : \mathsf{B}G \to \mathsf{B}H.$

(7.2.24)

In the first row, the stalk is the **filtered colimit** of all neighborhood sections, not the value on one chosen neighborhood.

Mac Lane and Moerdijk construct inverse-image sheaves through pulled-back étale spaces in [Mac Lane and Moerdijk(1992), sec. II.9], especially printed pp. 99–101.

In the second row, right actions are presheaves. Precomposition with $B\varphi^{\mathrm{op}}$ is restriction of actions. It has both Kan-extension adjoints, so it is the inverse-image part of the displayed geometric morphism. The homomorphism and geometric morphism point from H to G; restriction points from G-objects to H-objects.

Example 7.2.7 (internal inhabitation without a global point)
Work in $\mathbf{B}C_2$, the topos of right C_2-sets, and let $U = C_2$ carry the regular right action. The unique equivariant map $U \to 1$ is an epimorphism because its underlying function is surjective. In the internal language, this says that U is inhabited. Internal Group Actions are developed in [Mac Lane and Moerdijk(1992), sec. V.2, pp. 237–240].

A global element would be an equivariant map $1 \to U$. Its value would have to satisfy $u \cdot s = u$, but the regular action has no fixed point. Hence

$$\mathrm{Hom}_{\mathbf{B}C_2}(1, U) = \varnothing. \tag{7.2.25}$$

The two external tests are:

internal statement	external test
U is inhabited	$U \to 1$ is epi
U has a global point	$U^{C_2} \neq \varnothing$.

(7.2.26)

More generally, for a nontrivial group G, the regular right G-object $U_G = G$ satisfies

$$U_G \to 1 \text{ is epi}, \qquad \Gamma(U_G) = U_G^G = \varnothing. \tag{7.2.27}$$

Indeed, if $xg = x$ for every g, cancellation forces every g to be 1. Internal existence is therefore generalized or local existence; it does not choose a global invariant element. This distinction is not a physical existence claim.

Remark 7.2.8 (where the topos layer begins)
The finite tests separate calculations that merely take place in a topos from calculations that use topos-level structure:

test	lower level	topos value	verdict
C_3 action	linear action	none	fail
$C_2 \ltimes X$	isotropy	none	fail
C_2 torsor	free/transitive	classification	conditional
context	pullback/restriction	inverse image	conditional
U_G	no fixed point	internal inhabitation	strict.

(7.2.28)

Topoi enter these notes only when a result uses universal torsor classification, organizes several constructions as inverse image or base change, or relies on internal reasoning that differs from global sections. Ordinary Group Actions, Group Algebra calculations, and action groupoids stay below that layer.

This boundary also excludes several stronger claims. The spectrum of a center is not the original noncommutative Group Algebra. The action-groupoid example does not require a quotient stack. No site, descent theorem, or physical correspondence has been supplied by these finite tests. Each of those would need its own mathematical construction before extending the present layer.

7.3 Models and routes from binary tetrahedral structure

The concrete, internal, external, and abstract models of $2T$ retain different data. Their comparison opens a Group-Algebra route through the semidirect factors and a representation-theoretic route through the McKay correspondence.

Remark 7.3.1 (what the models of 2T retain)
The same binary tetrahedral symmetry now has several models. They answer different questions.

- The **concrete model** $T \leq \mathbb{H}^{\times}$ retains quaternion coordinates, multiplication, conjugation, and norm. It is the model for explicit calculations.
- The **internal semidirect-product structure** retains the same group T, but also marks the subgroups $Q, C \leq T$. The facts

$$T = QC, \qquad Q \trianglelefteq T, \qquad Q \cap C = \{1\} \tag{7.3.1}$$

  say that every element of T has a unique factorization qc. They expose how the two parts fit inside the concrete group.
- The **external model** $Q \rtimes_{\alpha} C$ replaces a quaternion by a pair (q, c). It retains the factors and records their interaction in the conjugation action $\alpha(c)(q) = cqc^{-1}$. Its multiplication is

$$(q, c)(q', c') = \big(q\alpha(c)(q'), cc'\big). \tag{7.3.2}$$

- An **abstract model** can specify the binary tetrahedral group without quaternion coordinates. This is convenient when only its group structure is needed, but coordinates and named factors must be restored by additional maps.

The multiplication map

$$\mu : Q \rtimes_\alpha C \longrightarrow T, \qquad (q, c) \longmapsto qc \tag{7.3.3}$$

is an isomorphism by the internal semidirect-product theorem. Indeed,

$$\mu\big((q, c)(q', c')\big) = q(cq'c^{-1})cc' = qcq'c' = \mu(q, c)\mu(q', c'). \tag{7.3.4}$$

Thus the external multiplication is not an analogy: it is exactly the concrete multiplication written in factor coordinates. The two carriers are still different, so a theorem or construction passes between them only through an explicit isomorphism. If it refers to the named factors or their action, the transport must also record the corresponding compatibility.

The internal and external constructions are developed in [Fré(2023), sec. 4.2.13, pp. 62–63] and [Isaev and Rubakov(2018), sec. 1.4.2, pp. 58–61]. The concrete identification of $2T$ with the Hurwitz unit group and $Q_8 \rtimes \mathbb{Z}/3\mathbb{Z}$ is given in [Voight(2021), sec. 11.2.4, p. 168].

Remark 7.3.2 (two routes from binary tetrahedral symmetry)
The semidirect-product model and the quaternion model open two different routes from the same group. One rewrites the Group Algebra in interacting factor data. The other places the group inside $\mathrm{SU}(2)$ and leads to the McKay graph.

7.3.2.1 The Group-Algebra route

Extend the conjugation action $\alpha : C \to \mathrm{Aut}(Q)$ linearly to real-algebra automorphisms

$$\overline{\alpha}_c : \mathbb{R}[Q] \longrightarrow \mathbb{R}[Q]. \tag{7.3.2.1.1}$$

On the vector space $\mathbb{R}[Q] \otimes_{\mathbb{R}} \mathbb{R}[C]$, write a pure tensor as $a \# c$ and define

$$(a \# c)(b \# d) = a\,\overline{\alpha}_c(b) \# cd, \tag{7.3.2.1.2}$$

extending bilinearly. This convention defines the **skew Group Algebra** $\mathbb{R}[Q] \rtimes_{\overline{\alpha}} C$; it is also a crossed product with trivial twisting cocycle.

The basis map

$$[q] \# c \longmapsto [qc] \tag{7.3.2.1.3}$$

respects multiplication by the calculation in what the models of 2T retain, and unique factorization makes it a bijection. Hence

$$\mathbb{R}[Q] \rtimes_{\overline{\alpha}} C \cong_{\mathbb{R}\text{-alg}} \mathbb{R}[T]. \tag{7.3.2.1.4}$$

This is the Group-Algebra form of $T \cong Q \rtimes_{\alpha} C$. The extension of group maps to Group-Algebra maps follows the convention in [Sengupta(2010), secs. 3.1–3.2, pp. 39–41].

A module over this skew Group Algebra can be read as an $\mathbb{R}[Q]$-module M together with a linear action of C satisfying

$$c \cdot (a \cdot m) = \overline{\alpha}_c(a) \cdot (c \cdot m). \tag{7.3.2.1.5}$$

The external model therefore exposes the compatibility needed to assemble representations from the two factors. It does not by itself decompose $\mathbb{R}[T]$ into simple blocks. That requires further representation-theoretic input.

7.3.2.2 The McKay route

Under the standard matrix realization of the unit quaternions as $\mathrm{SU}(2)$, the concrete group T becomes a finite subgroup of $\mathrm{SU}(2)$. Let τ_3 be its faithful 2-dimensional complex representation. The McKay graph has one vertex for each irreducible complex representation τ_i; the number of edges from τ_i to τ_j is the multiplicity of τ_j in

$$\tau_3 \otimes \tau_i. \tag{7.3.2.2.1}$$

For the binary tetrahedral group this graph is the affine Dynkin diagram $\widetilde{E}_6$. The character table, the seven irreducible representations, and the tensor-product calculation are given in [Stekolshchik(2008), table A.12, prop. A.10, and ex. A.11, pp. 172–174].

This graph connects the representation theory of $2T$ with quivers and the ADE classification. The same source relates a binary polyhedral subgroup $G \leq \mathrm{SU}(2)$ to the quotient $\mathbb{C}^2/G$, its invariant algebra, and its Kleinian singularity in [Stekolshchik(2008), secs. A.3–A.4, pp. 158–161]. These are routes toward quotient geometry and related orbifold constructions. They are not consequences of the semidirect-product theorem alone.

Neither route assigns physical meaning to Q, C, an algebra block, or a vertex of $\widetilde{E}_6$. They supply mathematical structures on which a physical correspondence could be stated and tested. A proposed correspondence must still identify the action, representation, preserved structure, and physical interpretation.

References

[Broué(2024)] Michel Broué. 2024. From Rings and Modules to Hopf Algebras: One Flew Over the Algebraist's Nest. (2024). (Cited in section 7.1.19)

[Cheng et al.(2019)] Tao Cheng, Huilan Li, and Yuping Yang. 2019. A New View of Generalized Clifford Algebras. *Advances in Applied Clifford Algebras* 29 (2019), 1–11. (Cited in section 7.1.16)

[Conlon(1964)] Samuel B Conlon. 1964. Twisted group algebras and their representations. *Journal of the Australian Mathematical Society* 4, 2 (1964), 152–173. (Cited in section 7.1.15)

[Etingof et al.(2011)] Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, and Elena Yudovina. 2011. *Introduction to Representation Theory*. Student Mathematical Library, Vol. 59. American Mathematical Society. https://math.mit.edu/~etingof/replect.pdf (Cited in section 6.1)

[Fré(2023)] Pietro Giuseppe Fré. 2023. *Discrete, Finite and Lie Groups: Comprehensive Group Theory in Geometry and Analysis*. De Gruyter. doi:10.1515/9783111201535 (Cited in sections 4.1, 4.2, 4.3, 4.4, and 7.3.1)

[Hamilton(2023)] Andrew JS Hamilton. 2023. The supergeometric algebra. *Advances in Applied Clifford Algebras* 33, 1 (2023), 12. https://arxiv.org/abs/2212.11998 (Cited in section 7.1.21)

[Hamilton and McMaken(2023)] Andrew JS Hamilton and Tyler McMaken. 2023. Unification of the four forces in the Spin (11, 1) geometric algebra. *Physica Scripta* 98, 8 (2023), 085306. https://arxiv.org/abs/2307.01243 (Cited in section 7.1.21)

[Isaev and Rubakov(2018)] Alexey P. Isaev and Valery A. Rubakov. 2018. *Theory of Groups and Symmetries: Finite Groups, Lie Groups, and Lie Algebras*. World Scientific. doi:10.1142/10898 (Cited in sections 4.4 and 7.3.1)

[James and Liebeck(2001)] Gordon Douglas James and Martin W Liebeck. 2001. *Representations and characters of groups*. Cambridge university press. (Cited in sections 7.1.3, 7.1.5, 7.1.6, 7.1.8, and 7.1.12)

[Lux and Pahlings(2010)] Klaus Lux and Herbert Pahlings. 2010. *Representations of groups: a computational approach*. Vol. 124. Cambridge University Press. (Cited in sections 7.1.5, 7.1.7, 7.1.9, 7.1.10, 7.1.11, 7.1.14, and 7.1.17)

[Mac Lane and Moerdijk(1992)] Saunders Mac Lane and Ieke Moerdijk. 1992. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Springer-Verlag, New York. doi:10.1007/978-1-4612-0927-0 (Cited in sections 7.2.4, 7.2.5, 7.2.6, and 7.2.7)

[Sengupta(2010)] Ambar N. Sengupta. 2010. Representations of Algebras and Finite Groups: An Introduction. (May 2010). https://math.uchicago.edu/~margalit/repthy/sengupta%2C%20finitegrouprepresent.pdf Course manuscript. (Cited in sections 1, 5.2, 5.3, 5.4, 5.5, 5.6, 6.1, 7.1.1, 7.1.2, 7.1.4, 7.1.18, and 7.3.2.1)

[Sims(1994)] Charles C Sims. 1994. *Computation with finitely presented groups*. Number 48. Cambridge University Press. (Cited in section 7.1.14)

[Stekolshchik(2008)] Rafael Stekolshchik. 2008. *Notes on Coxeter Transformations and the McKay Correspondence*. Springer Berlin, Heidelberg. doi:10.1007/978-3-540-77399-3 (Cited in section 7.3.2.2)

[Voight(2021)] John Voight. 2021. *Quaternion Algebras*. Graduate Texts in Mathematics, Vol. 288. Springer Cham. doi:10.1007/978-3-030-56694-4 (Cited in sections (document), 1, 2.1, 2.1, 2.2, 2.3, 2.4, 2.6, 3.2, 3.3, 3.4, 3.5, 4.5, 4.6, 4.8, 4.8, 6.1, 7.2.2, and 7.3.1)

[Webb(2007)] Peter Webb. 2007. Finite Group Representations for the Pure Mathematician. (December 2007). https://www.math.uchicago.edu/~margalit/repthy/Webb%2C%20Finite%20group%20reps%20for%20the%20pure%20mathematician.pdf Course manuscript, December 12, 2007. (Cited in sections 5.2, 5.4, and 5.5)

[Wilson(2021)] Robert A. Wilson. 2021. Finite symmetry groups in physics. *arXiv preprint arXiv:2102.02817* (2021). doi:10.48550/arXiv.2102.02817 (Cited in sections 1 and 2.6)

[Wilson(2024)] Robert A Wilson. 2024. A discrete model for Gell-Mann matrices. *arXiv preprint arXiv:2401.13000* (2024). https://arxiv.org/pdf/2401.13000 (Cited in section 7.1.21)

[Woit(2024)] Peter Woit. 2024. Quantum Theory, Groups and Representations: An Introduction. https://www.math.columbia.edu/~woit/QMbook/ Revised and expanded manuscript, January 9, 2024. (Cited in section 3.1)

Alphabetical Index