Notes on Clifford Algebras

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Readers may also check out Compendium on Spin groups.

1 Preliminaries

This section introduces the algebraic environment of Clifford Algebra, covering vector spaces, groups, algebras, representations, modules, multilinear algebras, quadratic forms, filtrations and graded algebras.

The material in this section should be familiar to the readers, but it is worth reading through it to become familiar with the notation and terminology that is used, as well as their counterparts in Lean, which usually require some additional treatment, both mathematically and technically (probably applicable to other formal proof verification systems).

Details can be found in the references in corresponding section, or you may use the L N button to check the corresponding Mathlib document and Lean 4 source code.

In this section, we follow [Jadczyk(2019)], with supplements from [Garling(2011)][Chen(2016)], and extensive modifications to match the counterparts in Lean's Mathlib.

1.1 Basics: from groups to modules

Convention 1.1.1 (Definition style)

In this document, we unify the informal mathematical language for a definition to:

Let `X` be a `concept X`.

A `concept Z` is a set/pair/triple/tuple $(Z, \text{op}, ...)$, satisfying:

1. `Z` is a `concept Y` over `X` under `op` .
2. `formula` for all `elements in Z` (`property`).
3. for each `element` in `concept X`, there exists `element` such that `formula` for all `elements in concept Z`.
4. op is called `op name`, for all `elements in Z`, we have
   (a) `formula`
   (b) `formula`

   (`property`).

By default, `X` is a set, `op` is a binary operation on `X`.

Definition 1.1.2 (Group [Garling(2011), 1.1])
A **group** is a pair $(G, *)$, satisfying:

1. $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$ (**associativity**).
2. there exists $1 \in G$ such that

$$1 * a = a * 1 = a \tag{1.1.3}$$

   for all $a \in G$.
3. for each $a \in G$, there exists $a^{-1} \in G$ such that

$$a * a^{-1} = a^{-1} * a = 1 \tag{1.1.4}$$

   .

Remark 1.1.5

It then follows that 1, the **identity** element, is unique, and that for each $g \in G$ the **inverse** g^{-1} is unique.

A group G is **abelian**, or **commutative**, if $g * h = h * g$ for all $g, h \in G$.

Notation 1.1.6 (Product)

In literatures, the binary operation of a group is usually called a product. It's denoted by juxtaposition gh, and is understood to be a mapping $(g, h) \mapsto g * h$ from $G \times G$ to G.

Here we use an explicit symbol for the operation. It can be denoted multiplicatively as $*$ in Group or additively as $+$ in AddGroup, where the identity element will be denoted by 1 or 0, respectively.

Sometimes we use notations with subscripts (e.g. $*_G$, 1_G) to indicate where they are.

Mathlib encodes the mapping $G \times G \to G$ as $G \to G \to G$, it is understood to be $G \to (G \to G)$, that sends $g \in G$ to a mapping that sends $h \in G$ to $g * h \in G$.

Furthermore, a mathematical construct, e.g. a group, is represented by a "type", as Lean has a dependent type theory foundation, see [Carneiro(2019)] and [Ullrich(2023), sec. 3.2].

Definition 1.1.7 (Monoid)

A **monoid** is a pair $(R, *)$, satisfying:

1. $(a * b) * c = a * (b * c)$ for all $a, b, c \in R$ (**associativity**).
2. there exists an element $1 \in R$ such that $1 * a = a * 1 = a$ for all $a \in R$
   i.e. 1 is the multiplicative identity (**neutral element**).

Definition 1.1.8 (Ring [Jadczyk(2019), 1.1])

A **ring** is a triple $(R, +, *)$, satisfying:

1. R is a **commutative group** under $+$.
2. R is a **monoid** under $*$.
3. for all $a, b, c \in R$, we have
   (a) $a * (b + c) = a * b + a * c$

(b) $(a+b)*c = a*c + b*c$

(left and right **distributivity** over $+$).

Remark 1.1.9

In applications to Clifford algebras R will be always assumed to be **commutative**.

Definition 1.1.10 (Division ring)

A **division ring** is a ring $(R, +, *)$, satisfying:

1. R contains at least 2 elements.
2. for all $a \neq 0$ in R, there exists a multiplicative inverse $a^{-1} \in R$ such that

$$a * a^{-1} = a^{-1} * a = 1 \tag{1.1.11}$$

Definition 1.1.12 (Module [Jadczyk(2019), 1.3])

Let R be a commutative ring. A **module** over R, called an R**-module**, is a pair $(M, \bullet)$, satisfying:

1. M is a group under $+$.
2. $\bullet : R \to M \to M$ is called **scalar multiplication**, for every $a, b \in R$, $x, y \in M$, we have
   (a) $a \bullet (x + y) = a \bullet x + b \bullet y$
   (b) $(a + b) \bullet x = a \bullet x + b \bullet x$
   (c) $a * (b \bullet x) = (a * b) \bullet x$
   (d) $1_R \bullet x = x$

Remark 1.1.13

The notation of scalar multiplication is generalized as heterogeneous scalar multiplication HMul in Mathlib:

$$\bullet : \alpha \to \beta \to \gamma \tag{1.1.14}$$

where α, β, γ are different types.

Definition 1.1.15 (Vector space [Jadczyk(2019), 1.5])
If R is a **division ring**, then a module M over R is called a **vector space**.

Remark 1.1.16
For generality, Mathlib uses Module throughout for vector spaces, particularly, for a vector space V, it's usually declared as

```
    Let $K$ be a division ring, a module $V$ over $K$ is a vector
    space
    where being a module requires $V$ to be a commutative group
    over $+$.
-/
variable [DivisionRing K] [AddCommGroup V] [Module K V]
```

For definitions/theorems about it, and most of them can be found under `Mathlib.LinearAlgebra` e.g. LinearIndependent.

Definition 1.1.17 (Submodule)
A **submodule** N of M is a module N such that every element of N is also an element of M.

If M is a vector space then N is called a **subspace**.

Definition 1.1.18 (Dual module)
The **dual module** $M^* : M \to_{l[R]} R$ is the R-module of all linear maps from M to R.

1.2 Algebras

Definition 1.2.1 (Ring homomorphism [Chen(2016), 4.5.1])
Let $(\alpha, +_\alpha, *_\alpha)$ and $(\beta, +_\beta, *_\beta)$ be rings.

A **ring homomorphism** from α to β is a map $\mathit{1} : \alpha \to_{+*} \beta$ such that

1. $\mathit{1}(x +_\alpha y) = \mathit{1}(x) +_\beta \mathit{1}(y)$ for each $x, y \in \alpha$.
2. $\mathit{1}(x *_\alpha y) = \mathit{1}(x) *_\beta \mathit{1}(y)$ for each $x, y \in \alpha$.
3. $\mathit{1}(1_\alpha) = 1_\beta$.

Definition 1.2.2 (Isomorphism, endomorphism, automorphism)

Isomorphism $A \cong B$ is a bijective **homomorphism** $\phi : A \to B$ (it follows that $\phi^{-1} : B \to A$ is also a **homomorphism**).

Endomorphism is a **homomorphism** from an object to itself, denoted $\mathrm{End}(A)$.

Automorphism is an **endomorphism** which is also an **isomorphism**, denoted $\mathrm{Aut}(A)$.

Definition 1.2.3 (Algebra)

Let R be a commutative ring. An **algebra** A over R is a pair $(A, \bullet)$, satisfying:

1. A is a **ring** under $*$.
2. there exists a **ring homomorphism** from R to A, denoted $\mathit{1} : R \to_{+*} A$.
3. $\bullet : R \to M \to M$ is a **scalar multiplication**
4. for every $r \in R$, $x \in A$, we have
   (a) $r * x = x * r$ (commutativity between R and A)
   (b) $r \bullet x = r * x$ (definition of scalar multiplication)

   where we omitted that the ring homomorphism $\mathit{1}$ is applied to r before each multiplication.

Notation 1.2.4

Following literatures, for $r \in R$, usually we write $\mathit{1}_A(r) : R \to_{+*} A$ as a product $r\,\mathit{1}_A$ if not omitted, while they are written as a call to `algebraMap _ _ r` in Mathlib, which is defined to be `Algebra.toRingHom r`.

Remark 1.2.5

The definition above (adopted in Mathlib) is more general than the definition in literature (e.g. [Jadczyk(2019), 1.6]):

Let R be a commutative ring. An **algebra** A over R is a pair $(M, *)$, satisfying:

1. A is a **module** M over R under $+$ and $\bullet$.
2. A is a **ring** under $*$.
3. For $x, y \in A, a \in R$, we have

$$a \bullet (x * y) = (a \bullet x) * y = x * (a \bullet y) \tag{1.2.6}$$

See *Implementation notes* in Algebra for details.

Remark 1.2.7

What's simply called algebra is actually associative algebra with identity, a.k.a. **associative unital algebra**. See the red herring principle for more about such phenomenon for naming, particularly the example of (possibly) **nonassociative algebra**.

Definition 1.2.8 (Algebra homomorphism)

Let A and B be R-algebras. 1_A and 1_B are **ring homomorphisms** from R to A and B, respectively.

A **algebra homomorphism** from A to B is a map $f : \alpha \to_a \beta$ such that

1. f is a **ring homomorphism**
2. $f(1_A(r)) = 1_B(r)$ for each $r \in R$

Definition 1.2.9 (Ring quotient)

Let R be a non-commutative ring, r an arbitrary equivalence relation on R. The **ring quotient** of R by r is the quotient of R by the strengthen equivalence relation of r such that for all a, b, c in R:

1. $a + c \sim b + c$ if $a \sim b$
2. $a * c \sim b * c$ if $a \sim b$
3. $a * b \sim a * c$ if $b \sim c$

i.e. the equivalence relation is compatible with the ring operations $+$ and $*$.

Remark 1.2.10

As ideals haven't been formalized for the non-commutative case, Mathlib uses RingQuot in places where the quotient of non-commutative rings by ideal is needed.

The universal properties of the quotient are proven, and should be used instead of the definition that is subject to change.

Definition 1.2.11 (Free algebra)

Let X be an arbitrary set. An **free R-algebra** on X (or "**generated by** X"), named A, is the **ring quotient** of the following inductively constructed set $A_{\texttt{pre}}$

1. for all x in X, there exists a map $X \longrightarrow A_{\texttt{pre}}$.
2. for all r in R, there exists a map $R \longrightarrow A_{\texttt{pre}}$.
3. for all a, b in $A_{\texttt{pre}}$, $a + b$ is in $A_{\texttt{pre}}$.
4. for all a, b in $A_{\texttt{pre}}$, $a * b$ is in $A_{\texttt{pre}}$.

by that equivalence relation that makes A an R**-algebra**, namely:

1. there exists a **ring homomorphism** from R to $A_{\texttt{pre}}$, denoted $R \longrightarrow_{+*} A_{\texttt{pre}}$.
2. A is a **commutative group** under $+$.
3. A is a **monoid** under $*$.
4. left and right **distributivity** under $*$ over $+$.
5. $0 * a \sim a * 0 \sim 0$.
6. for all a, b, c in A, if $a \sim b$, we have
   (a) $a + c \sim b + c$
   (b) $c + a \sim c + b$
   (c) $a * c \sim b * c$
   (d) $c * a \sim c * b$

   (**compatibility** with the ring operations $+$ and $*$)

Remark 1.2.12

Similar to Remark 1.2.7, what we defined here is the **free (associative, unital) R-algebra** on X, it can be denoted $R\langle X\rangle$, expressing that it's freely generated by R and X, where X is the set of generators.

Definition 1.2.13 (Linear map)

Let R, S be rings, M an R-module, N an S-module. A **linear map** from M to N is a function $f : M \to_l N$ over a ring homomorphism $\sigma : R \to_{+*} S$, satisfying:

1. $f(x+y) = f(x) + f(y)$ for all $x, y \in M$.
2. $f(r \bullet x) = \sigma(r) \bullet f(x)$ for all $r \in R, x \in M$.

Remark 1.2.14 (Lin)

The set of all linear maps from M to M' is denoted $\mathrm{Lin}(M, M')$, and $\mathrm{Lin}(M)$ for mapping from M to itself.

$\mathrm{Lin}(M)$ is an **endomorphism**.

Definition 1.2.15 (Tensor algebra)

Let A be a **free** R**-algebra** generated by module M, let $\iota : M \to A$ denote the map from M to A.

An **tensor algebra** over M (or "of M") T is the **ring quotient** of the **free** R**-algebra** generated by M, by the equivalence relation satisfying:

1. for all a, b in M, $\iota(a+b) \sim \iota(a) + \iota(b)$.
2. for all r in R, a in M, $\iota(r \bullet a) \sim r * \iota(a)$.

i.e. making the inclusion of M into an R**-linear map**.

Remark 1.2.16

The definition above is equivalent to the following definition in literature (e.g. [Jadczyk(2019), 1.7]):

Let M be a module over R. An algebra T is called a **tensor algebra** over M (or "of M") if it satisfies the following universal properties:

1. T is an algebra containing M as a **submodule**, and it is **generated by** M,
2. Every linear mapping λ of M into an algebra A over R, can be extended to a **homomorphism** θ of T into A.

1.3 Forms

Definition 1.3.1 (Bilinear form)
Let R be a ring, M an R-module.

An **bilinear form** B over M is a map $B : M \to M \to R$, satisfying:

1. $B(x + y, z) = B(x, z) + B(y, z)$
2. $B(x, y + z) = B(x, y) + B(x, z)$
3. $B(a \bullet x, y) = a * B(x, y)$
4. $B(x, a \bullet y) = a * B(x, y)$

for all $a \in R, x, y, z \in M$.

Definition 1.3.2 (Quadratic form [Jadczyk(2019), 1.9])
Let R be a commutative ring, M a R-module.

An **quadratic form** Q over M is a map $Q : M \to R$, satisfying:

1. $Q(a \bullet x) = a * a * Q(x)$ for all $a \in R, x \in M$.
2. there exists a companion **bilinear form** $B : M \to M \to R$, such that $Q(x + y) = Q(x) + Q(y) + B(x, y)$

In [Jadczyk(2019)], the bilinear form is denoted Φ, and called the **polar form** associated with the quadratic form Q, or simply the polar form of Q.

Remark 1.3.3

This notion generalizes to commutative semirings using the approach in [Izhakian et al.(2016)].

2 Clifford Algebra

Here we provide a detailed account of the formalization of Clifford Algebra [Hestenes and Sobczyk(1984)] in the Lean 4 theorem prover and programming language [Moura and Ullrich(2021)][de Moura et al.(2015)][Ullrich(2023)] and using its Mathematical Library Mathlib [The mathlib Community(2020)].

The primary references of the formalization are [Wieser and Song(2022)] and [Wieser(2024)]. This section and the previous section are adapted from our [Wieser and Song(2024)].

2.1 Definition

Definition 2.1.1 (Clifford algebra [Wieser and Song(2022)])

Let M be a module over a commutative ring R, equipped with a quadratic form $Q : M \to R$.

Let $\iota : M \to_{l[R]} T(M)$ be the canonical R-linear map for the tensor algebra $T(M)$.

Let $\mathit{1} : R \to_{+*} T(M)$ be the canonical map from R to $T(M)$, as a ring homomorphism.

A **Clifford algebra** over Q, denoted $\mathcal{C}\ell(Q)$, is the **ring quotient** of the **tensor algebra** $T(M)$ by the equivalence relation satisfying $\iota(m)^2 \sim \mathit{1}(Q(m))$ for all $m \in M$.

The natural quotient map is denoted $\pi : T(M) \to \mathcal{C}\ell(Q)$ in some literatures, or π_Φ/π_Q to emphasize the bilinear form Φ or the quadratic form Q, respectively.

Remark 2.1.2

In literatures, M is often written V, and the quotient is taken by the two-sided ideal I_Q generated from the set $\{v \otimes v - Q(v) | v \in V\}$. See also Clifford algebra [pr-spin].

As of writing, Mathlib does not have direct support for two-sided ideals, but it does support the equivalent operation of taking the **ring quotient** by a suitable closure of a relation like $v \otimes v \sim Q(v)$.

Hence the definition above.

Remark 2.1.3

This definition and what follows in Mathlib is initially presented in [Wieser and Song(2022)], some further developments are based on [Grinberg(2016)], and in turn based on [Bourbaki(2007)] which is in French and never translated to English.

The most informative English reference on [Bourbaki(2007)] is [Jadczyk(2019)], which has an updated exposition in [Jadczyk(2023)].

Example 2.1.4 (Clifford algebra over a vector space)

Let V be a vector space $\mathbb{R}^n$ over $\mathbb{R}$, equipped with a quadratic form Q.

Since $\mathbb{R}$ is a commutative ring and V is a module, the definition above applies.

Definition 2.1.5 (Clifford map [Wieser and Song(2022)])

We denote the canonical R-linear map to the Clifford algebra $\mathcal{C}\ell(M)$ by $\iota : M \to_{l[R]} \mathcal{C}\ell(M)$.

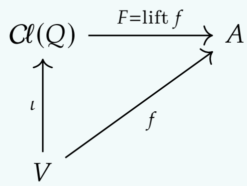
It's denoted i_Φ or just i in some literatures.

Definition 2.1.6 (Clifford lift [Wieser and Song(2022)])

Given a linear map $f : M \to_{l[R]} A$ into an R-algebra A, satisfying $f(m)^2 = Q(m)$ for all $m \in M$, called **is Clifford**, the canonical **lift** of f is defined to be a **algebra homomorphism** from $\mathcal{C}\ell(Q)$ to A, denoted lift $f : \mathcal{C}\ell(Q) \to_a A$.

Theorem 2.1.7 (Universal property [Wieser and Song(2022)])
Given $f : M \to_{l[R]} A$, which **is Clifford**, $F = \text{lift}\, f$ (denoted $\bar{f}$ in some literatures), we have:

1. $F \circ \iota = f$, i.e. the following diagram commutes:



2. lift is unique, i.e. given $G : C\ell(Q) \to_a A$, we have:

$$G \circ \iota = f \iff G = \text{lift}\, f \tag{2.1.8}$$

Remark 2.1.9
The universal property of the Clifford algebras is now proven, and should be used instead of the definition that is subject to change.

Definition 2.1.10 (Exterior algebra [Wieser and Song(2022)])
An **Exterior algebra** over M is the Clifford algebra $C\ell(Q)$ where $Q(m) = 0$ for all $m \in M$.

2.2 Operations

Convention 2.2.1 ([Wieser and Song(2022)])
Same as the previous section, let M be a module over a commutative ring R, equipped with a quadratic form $Q : M \to R$.

We also use m or $m_1, m_2, \ldots$ to denote elements of M, i.e. vectors, and x, y, z to denote elements of $C\ell(Q)$.

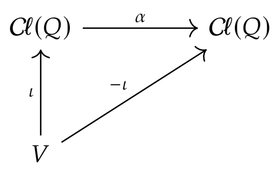
Definition 2.2.2 (Grade involution [Wieser and Song(2022)])
Grade involution, intuitively, is negating each basis vector.

Formally, it's an **algebra homomorphism** $\alpha : C\ell(Q) \to_a C\ell(Q)$, satisfying:

1. $\alpha \circ \alpha = \mathrm{id}$
2. $\alpha(\iota(m)) = -\iota(m)$

for all $m \in M$.

That is, the following diagram commutes:



It's called **main involution** α or **main automorphism** in [Jadczyk(2019)], the **canonical automorphism** in [Gallier(2008)].

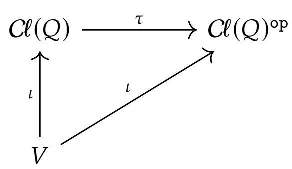
It's denoted $\hat{m}$ in [Lounesto(2001)], $\alpha(m)$ in [Jadczyk(2019)], m^* in [Chisolm(2012)].

Definition 2.2.3 (Grade reversion [Wieser and Song(2022)])
Grade reversion, intuitively, is reversing the multiplication order of basis vectors.

Formally, it's an **algebra homomorphism** $\tau : C\ell(Q) \to_a C\ell(Q)^{\mathrm{op}}$, satisfying:

1. $\tau(m_1 m_2) = \tau(m_2)\tau(m_1)$
2. $\tau \circ \tau = \mathrm{id}$
3. $\tau(\iota(m)) = \iota(m)$

That is, the following diagram commutes:



It's called **anti-involution** τ in [Jadczyk(2019)], the **canonical anti-automorphism** in [Gallier(2008)], also called **transpose**/**transposition** in some literature, following tensor algebra or matrix.

It's denoted $\tilde{m}$ in [Lounesto(2001)], m^τ in [Jadczyk(2019)] (with variants like m^t or $m^\top$ in other literatures), $m^\dagger$ in [Chisolm(2012)].

Definition 2.2.4 (Clifford conjugate [Wieser and Song(2022)])

Clifford conjugate is an **algebra homomorphism** $* : \mathcal{C}\ell(Q) \to_a \mathcal{C}\ell(Q)$, denoted x^* (or even $x^\dagger$, x^v in some literatures), defined to be:

$$x^* = \text{reverse}(\text{involute}(x)) = \tau(\alpha(x)) \tag{2.2.5}$$

for all $x \in \mathcal{C}\ell(Q)$, satisfying (as a **$*$-ring**):

1. $(x + y)^* = (x)^* + (y)^*$
2. $(xy)^* = (y)^*(x)^*$
3. $* \circ * = \text{id}$
4. $1^* = 1$

and (as a **$*$-algebra**):

$$(rx)^* = r'x^* \tag{2.2.6}$$

for all $r \in R$, $x, y \in \mathcal{C}\ell(Q)$ where $'$ is the involution of the commutative $*$-ring R.

Note: In our current formalization in Mathlib, the application of the involution on r is ignored, as there appears to be nothing in the literature that advocates doing this.

Clifford conjugate is denoted $\bar{m}$ in [Lounesto(2001)] and most literatures, $m^\ddagger$ in [Chisolm(2012)].

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